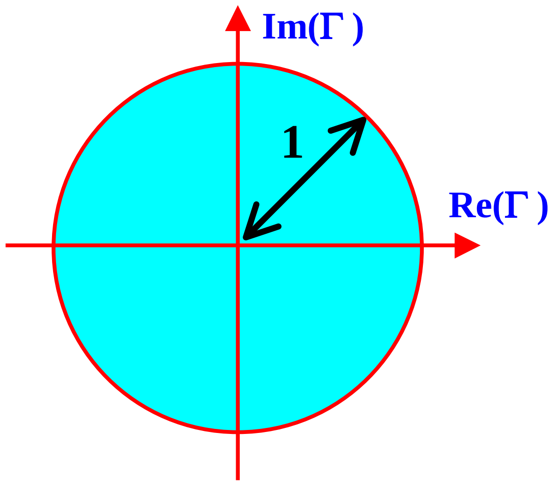
Smith Chart

The Smith chart is one of the most useful graphical tools for high frequency circuit applications. The chart provides a clever way to visualize complex functions and it continues to endure popularity decades after its original conception.

From a mathematical point of view, the Smith chart is simply a representation of all possible complex impedances with respect to coordinates defined by the reflection coefficient.



The domain of definition of the reflection coefficient is a circle of radius 1 in the complex plane. This is also the domain of the Smith chart.

The goal of the Smith chart is to identify all possible impedances on the domain of existence of the reflection coefficient. To do so, we start from the general definition of line impedance (which is equally applicable to the load impedance)

$$Z(d) = \frac{V(d)}{I(d)} = Z_0 \frac{1+\Gamma(d)}{1-\Gamma(d)}$$

This provides the complex function $Z(d) = f\{\mathrm{Re}(\Gamma), \mathrm{Im}(\Gamma)\}$ that we want to graph. It is obvious that the result would be applicable only to lines with exactly characteristic impedance Z_0.

In order to obtain universal curves, we introduce the concept of normalized impedance

$$z(d) = \frac{Z(d)}{Z_0} = \frac{1+\Gamma(d)}{1-\Gamma(d)}$$

The normalized impedance is represented on the Smith chart by using families of curves that identify the normalized resistance *r* (real part) and the normalized reactance *x* (imaginary part)

$$z(d) = \mathrm{Re}(z) + j\,\mathrm{Im}(z) = r + jx$$

Let's represent the reflection coefficient in terms of its coordinates

$$\Gamma(d) = \mathrm{Re}(\Gamma) + j\,\mathrm{Im}(\Gamma)$$

Now we can write

$$r + jx = \frac{1 + \mathrm{Re}(\Gamma) + j\,\mathrm{Im}(\Gamma)}{1 - \mathrm{Re}(\Gamma) - j\,\mathrm{Im}(\Gamma)}$$

$$= \frac{1 - \mathrm{Re}^2(\Gamma) - \mathrm{Im}^2(\Gamma) + j2\,\mathrm{Im}(\Gamma)}{(1 - \mathrm{Re}(\Gamma))^2 + \mathrm{Im}^2(\Gamma)}$$

The real part gives

$$r = \frac{1 - \mathrm{Re}^2(\Gamma) - \mathrm{Im}^2(\Gamma)}{\left(1 - \mathrm{Re}(\Gamma)\right)^2 + \mathrm{Im}^2(\Gamma)}$$

Add a quantity equal to zero

$$r\left(\mathrm{Re}(\Gamma) - 1\right)^2 + \left(\mathrm{Re}^2(\Gamma) - 1\right) + r\,\mathrm{Im}^2(\Gamma) + \mathrm{Im}^2(\Gamma) + \underbrace{\frac{1}{1+r} - \frac{1}{1+r}}_{=\,0} = 0$$

$$\left[r\left(\mathrm{Re}(\Gamma) - 1\right)^2 + \left(\mathrm{Re}^2(\Gamma) - 1\right) + \frac{1}{1+r}\right] + (1+r)\,\mathrm{Im}^2(\Gamma) = \frac{1}{1+r}$$

$$(1+r)\left[\mathrm{Re}^2(\Gamma) - 2\,\mathrm{Re}(\Gamma)\frac{r}{1+r} + \frac{r^2}{(1+r)^2}\right] + (1+r)\,\mathrm{Im}^2(\Gamma) = \frac{1}{1+r}$$

$$\Rightarrow \qquad \left[\mathrm{Re}(\Gamma) - \frac{r}{1+r}\right]^2 + \mathrm{Im}^2(\Gamma) = \left(\frac{1}{1+r}\right)^2$$

Equation of a circle

The imaginary part gives

$$x = \frac{2\,\mathrm{Im}(\Gamma)}{(1-\mathrm{Re}(\Gamma))^2 + \mathrm{Im}^2(\Gamma)}$$

Multiply by x and add a quantity equal to zero

$$x^2\left[(1-\mathrm{Re}(\Gamma))^2 + \mathrm{Im}^2(\Gamma)\right] - 2x\,\mathrm{Im}(\Gamma) + \underbrace{1-1}_{=0} = 0$$

$$\left[(1-\mathrm{Re}(\Gamma))^2 + \mathrm{Im}^2(\Gamma)\right] - \frac{2}{x}\mathrm{Im}(\Gamma) + \frac{1}{x^2} = \frac{1}{x^2}$$

$$(1-\mathrm{Re}(\Gamma))^2 + \left[\mathrm{Im}^2(\Gamma) - \frac{2}{x}\mathrm{Im}(\Gamma) + \frac{1}{x^2}\right] = \frac{1}{x^2}$$

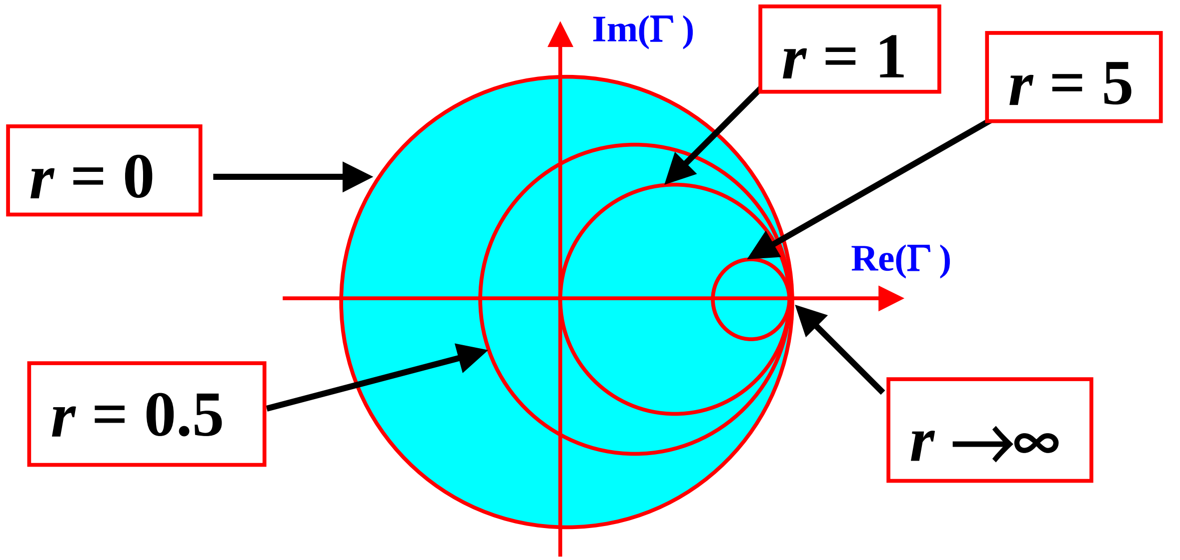
$$\Rightarrow \qquad (\mathrm{Re}(\Gamma)-1)^2 + \left[\mathrm{Im}(\Gamma) - \frac{1}{x}\right]^2 = \frac{1}{x^2}$$

Equation of a circle

The result for the real part indicates that on the complex plane with coordinates (Re(Γ), Im(Γ)) all the possible impedances with a given normalized resistance r are found on a circle with

$$\text{Center} = \left\{ \frac{r}{1+r}, 0 \right\} \qquad \text{Radius} = \frac{1}{1+r}$$

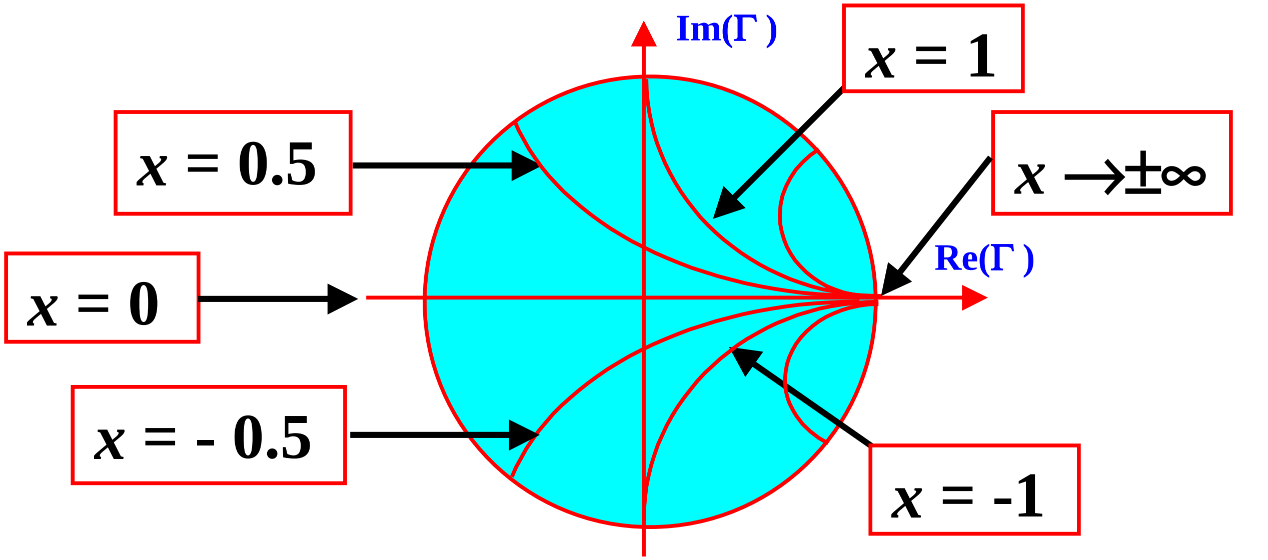
As the normalized resistance r varies from 0 to ∞ , we obtain a family of circles completely contained inside the domain of the reflection coefficient $|\,\Gamma\,| \leq 1$.



The result for the imaginary part indicates that on the complex plane with coordinates (Re(Γ), Im(Γ)) all the possible impedances with a given normalized reactance x are found on a circle with

$$\text{Center} = \left\{ 1 , \frac{1}{x} \right\} \qquad \text{Radius} = \frac{1}{x}$$

As the normalized reactance x varies from -∞ to ∞ , we obtain a family of arcs contained inside the domain of the reflection coefficient $|\Gamma| \leq 1$.



Basic Smith Chart techniques for loss-less transmission lines

- Given $Z(\mathrm{d})$ ⇒ Find $\Gamma(\mathrm{d})$
  Given $\Gamma(\mathrm{d})$ ⇒ Find $Z(\mathrm{d})$

- Given Γ_R and Z_R ⇒ Find $\Gamma(\mathrm{d})$ and $Z(\mathrm{d})$
  Given $\Gamma(\mathrm{d})$ and $Z(\mathrm{d})$ ⇒ Find Γ_R and Z_R

- Find d_{max} and d_{min} (maximum and minimum locations for the voltage standing wave pattern)

- Find the Voltage Standing Wave Ratio (VSWR)

- Given $Z(\mathrm{d})$ ⇒ Find $Y(\mathrm{d})$
  Given $Y(\mathrm{d})$ ⇒ Find $Z(\mathrm{d})$

Given $Z(\mathrm{d}) \Rightarrow$ Find $\Gamma(\mathrm{d})$

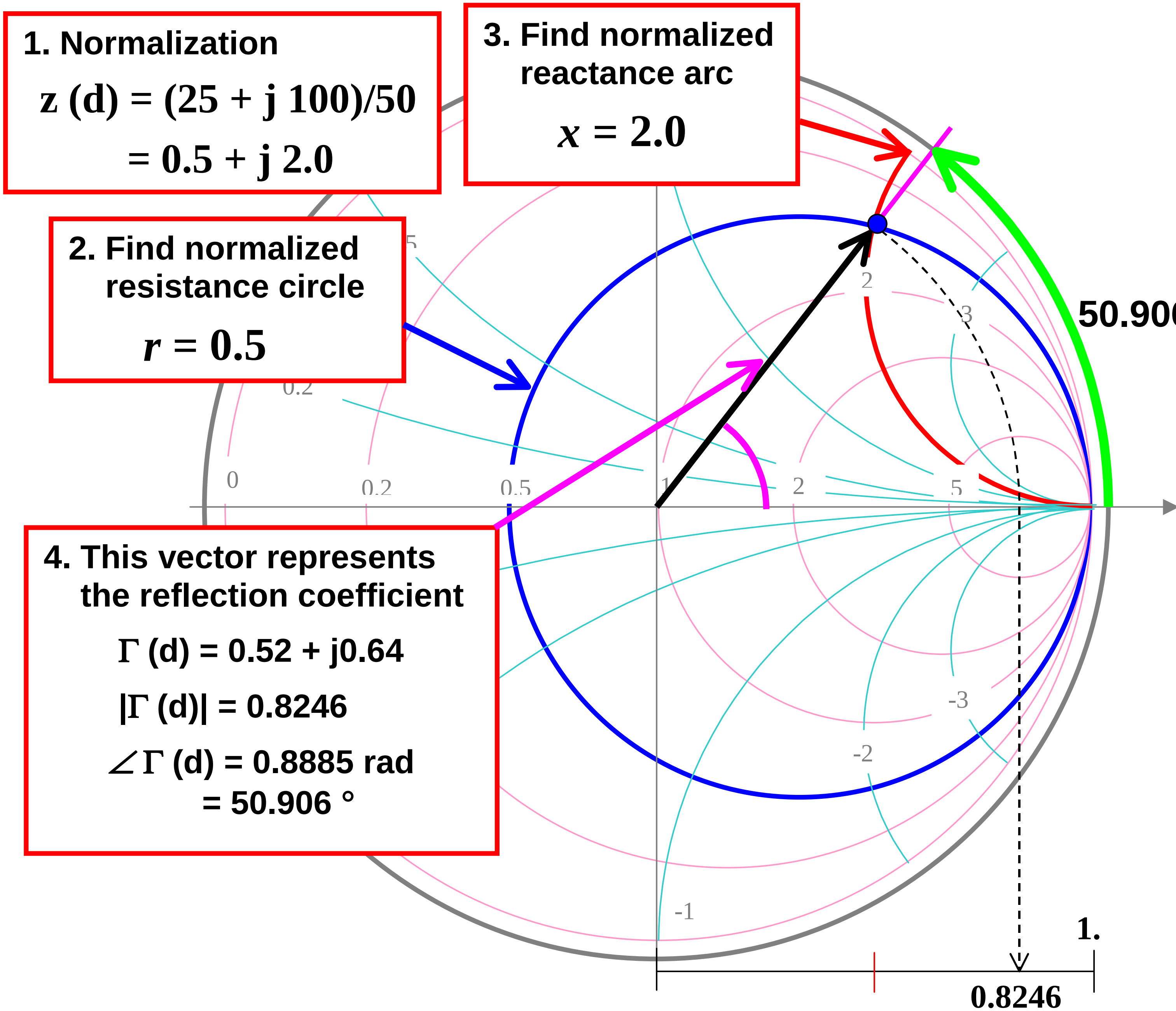
1. **Normalize the impedance**

$$z(\mathrm{d}) = \frac{Z(\mathrm{d})}{Z_0} = \frac{R}{Z_0} + j\,\frac{X}{Z_0} = r + j\,x$$

2. **Find the circle of constant normalized resistance r**
3. **Find the arc of constant normalized reactance x**
4. **The intersection of the two curves indicates the reflection coefficient in the complex plane. The chart provides directly the magnitude and the phase angle of $\Gamma(\mathrm{d})$**

Example: Find $\Gamma(\mathrm{d})$, given

$$Z(\mathrm{d}) = 25 + j\,100\ \Omega \qquad \text{with} \quad Z_0 = 50\ \Omega$$


1. Normalization
z (d) = (25 + j 100)/50
= 0.5 + j 2.0
2. Find normalized resistance circle
r = 0.5
3. Find normalized reactance arc
x = 2.0
4. This vector represents the reflection coefficient
Γ (d) = 0.52 + j0.64
|Γ (d)| = 0.8246
∠Γ (d) = 0.8885 rad
= 50.906 °
50.906 °
0
0.2
0.5
1
2
5
-1
-2
-3
3
1.
0.8246

Given $\Gamma(\mathrm{d})$ $\Rightarrow$ Find $Z(\mathrm{d})$

1. **Determine the complex point representing the given reflection coefficient $\Gamma(\mathrm{d})$ on the chart.**
2. **Read the values of the normalized resistance r and of the normalized reactance x that correspond to the reflection coefficient point.**
3. **The normalized impedance is**

$$z(\mathrm{d}) = r + j\,x$$

and the actual impedance is

$$Z(\mathrm{d}) = Z_0\; z(\mathrm{d}) = Z_0\,(r + j\,x) = Z_0\, r + j\, Z_0\, x$$

Given Γ_R and Z_R $\Longleftrightarrow$ Find $\Gamma(d)$ and $Z(d)$

NOTE: the magnitude of the reflection coefficient is constant along a loss-less transmission line terminated by a specified load, since

$$\left|\Gamma(d)\right| = \left|\Gamma_R \exp(-j2\beta d)\right| = \left|\Gamma_R\right|$$

Therefore, on the complex plane, a circle with center at the origin and radius $|\Gamma_R|$ represents all possible reflection coefficients found along the transmission line. When the circle of constant magnitude of the reflection coefficient is drawn on the Smith chart, one can determine the values of the line impedance at any location.

The graphical step-by-step procedure is:

1. **Identify the load reflection coefficient Γ_R and the normalized load impedance Z_R on the Smith chart.**

2. **Draw the circle of constant reflection coefficient amplitude $|\Gamma(d)| = |\Gamma_R|$.**
3. **Starting from the point representing the load, travel on the circle in the clockwise direction, by an angle**

$$\theta = 2\beta d = 2\frac{2\pi}{\lambda}d$$

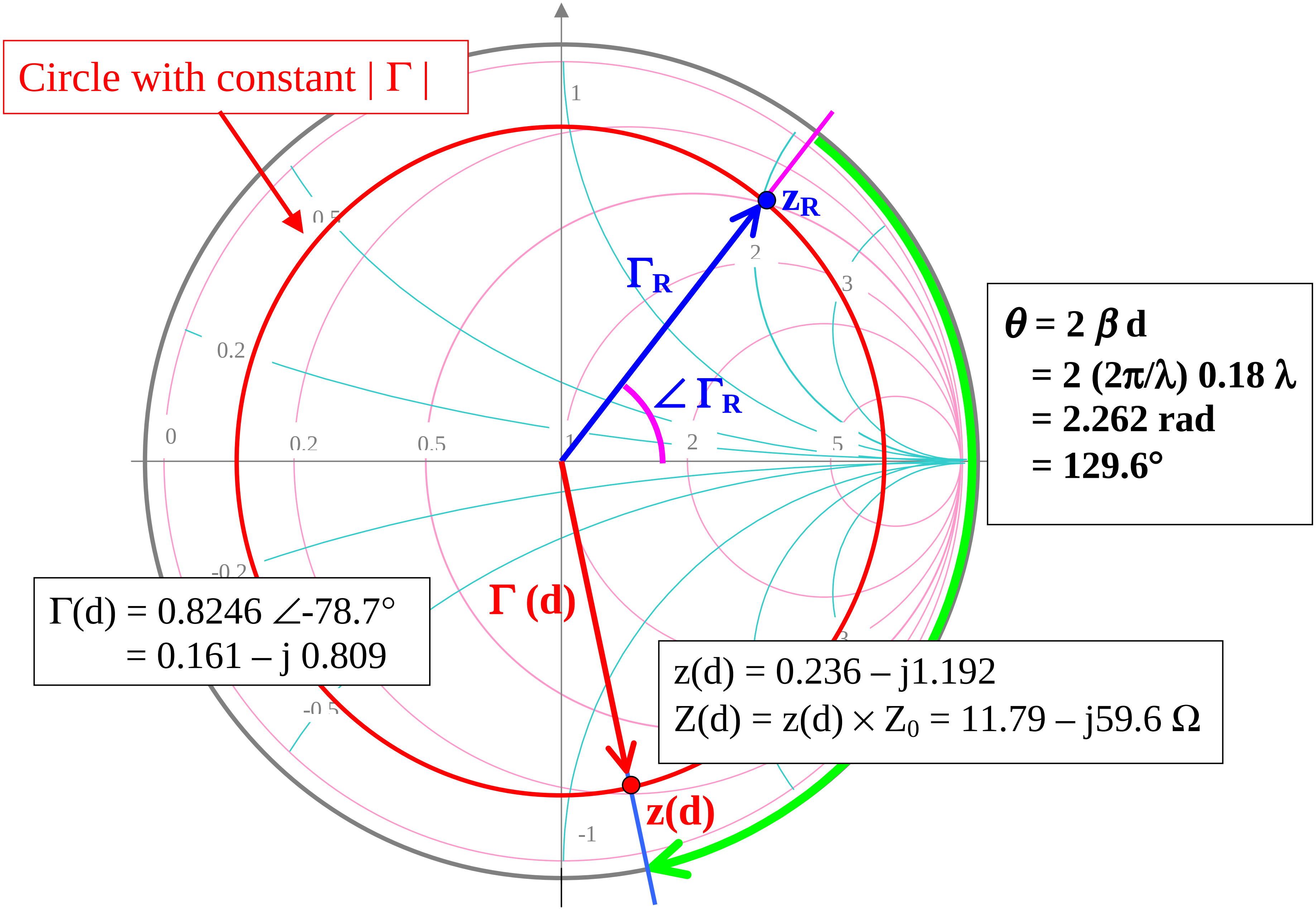
4. **The new location on the chart corresponds to location d on the transmission line. Here, the values of $\Gamma(d)$ and $Z(d)$ can be read from the chart as before.**

Example: Given

$$Z_R = 25 + j\,100\,\Omega \quad \text{with} \quad Z_0 = 50\,\Omega$$

find

$$Z(d) \quad \text{and} \quad \Gamma(d) \qquad \text{for} \qquad d = 0.18\lambda$$


Circle with constant | Γ |
z_R
Γ_R
$\angle \Gamma_R$
$\theta = 2\,\beta\,d$
$= 2\,(2\pi/\lambda)\,0.18\,\lambda$
$= 2.262$ rad
$= 129.6°$
Γ (d)
$\Gamma(d) = 0.8246 \angle -78.7°$
$= 0.161 - j\,0.809$
$z(d) = 0.236 - j1.192$
$Z(d) = z(d) \times Z_0 = 11.79 - j59.6\ \Omega$
z(d)

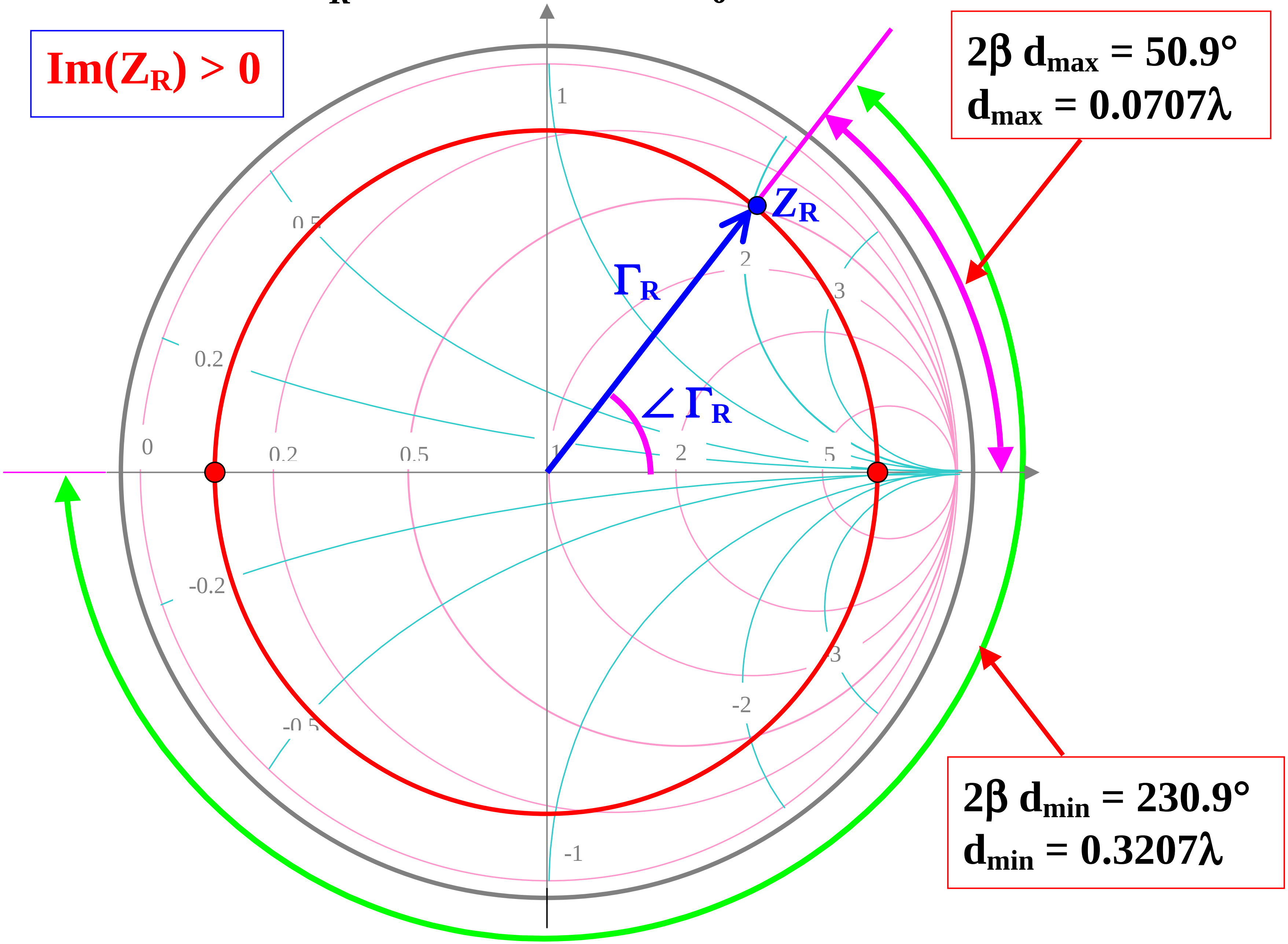
Given Γ_R and Z_R $\Rightarrow$ Find d_{max} and d_{min}

1. **Identify on the Smith chart the load reflection coefficient Γ_R or the normalized load impedance Z_R.**
2. **Draw the circle of constant reflection coefficient amplitude $|\Gamma(d)| = |\Gamma_R|$. The circle intersects the real axis of the reflection coefficient at two points which identify d_{max} (when $\Gamma(d)$ = Real positive) and d_{min} (when $\Gamma(d)$ = Real negative)**
3. **A commercial Smith chart provides an outer graduation where the distances normalized to the wavelength can be read directly. The angles, between the vector Γ_R and the real axis, also provide a way to compute d_{max} and d_{min}.**

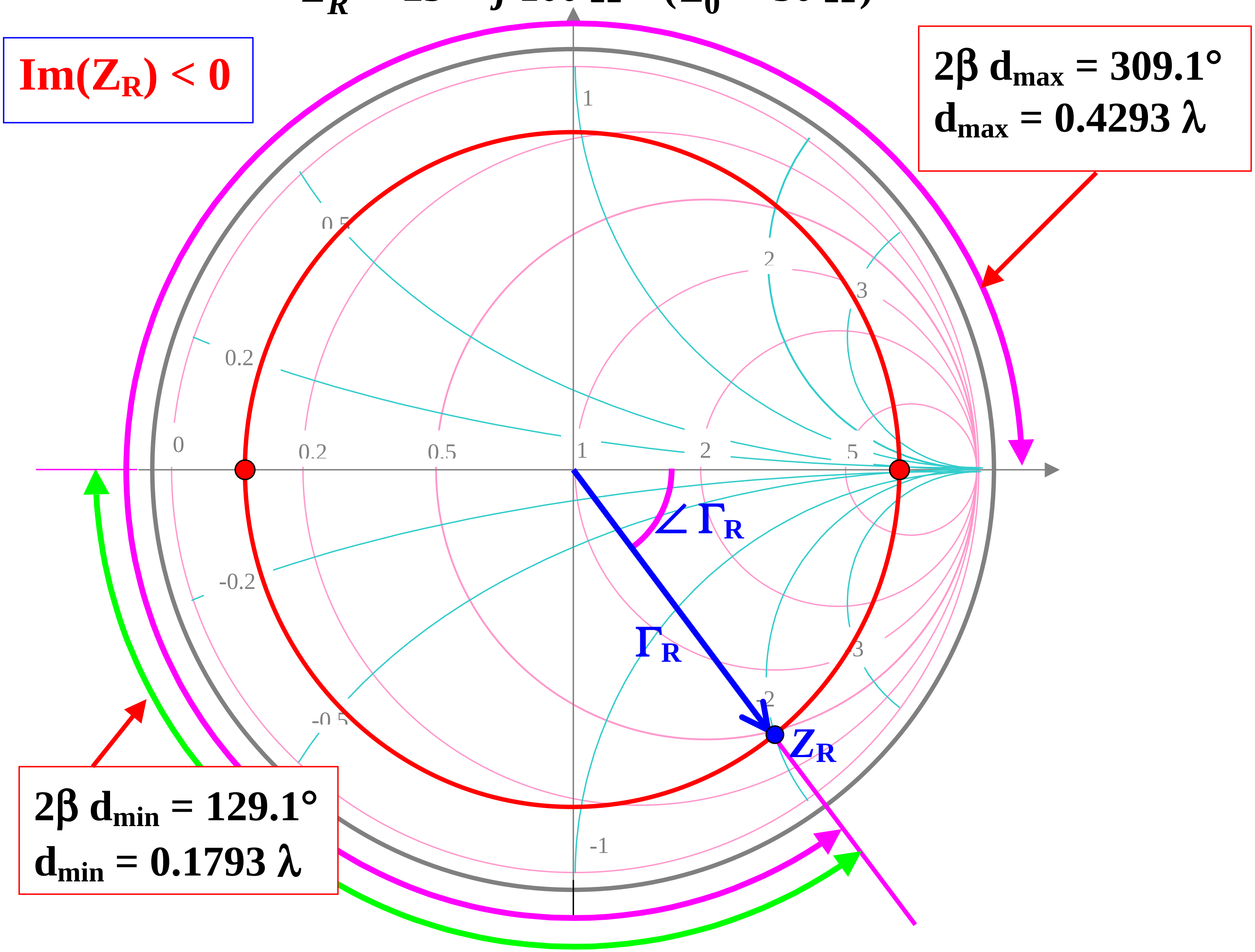
Example: Find d_{max} and d_{min} for

$$Z_R = 25 + j\,100\ \Omega \ ; \ Z_R = 25 - j100\Omega \quad (Z_0 = 50\ \Omega)$$

$$Z_R = 25 + j\,100\ \Omega \quad (Z_0 = 50\ \Omega)$$



$Z_R = 25 - j\,100\ \Omega \quad (Z_0 = 50\ \Omega)$



Given Γ_R and Z_R $\Rightarrow$ Find the Voltage Standing Wave Ratio (VSWR)

The Voltage standing Wave Ratio or VSWR is defined as

$$VSWR = \frac{V_{max}}{V_{min}} = \frac{1+|\Gamma_R|}{1-|\Gamma_R|}$$

The normalized impedance at a maximum location of the standing wave pattern is given by

$$z(d_{max}) = \frac{1+\Gamma(d_{max})}{1-\Gamma(d_{max})} = \frac{1+|\Gamma_R|}{1-|\Gamma_R|} = VSWR!!!$$

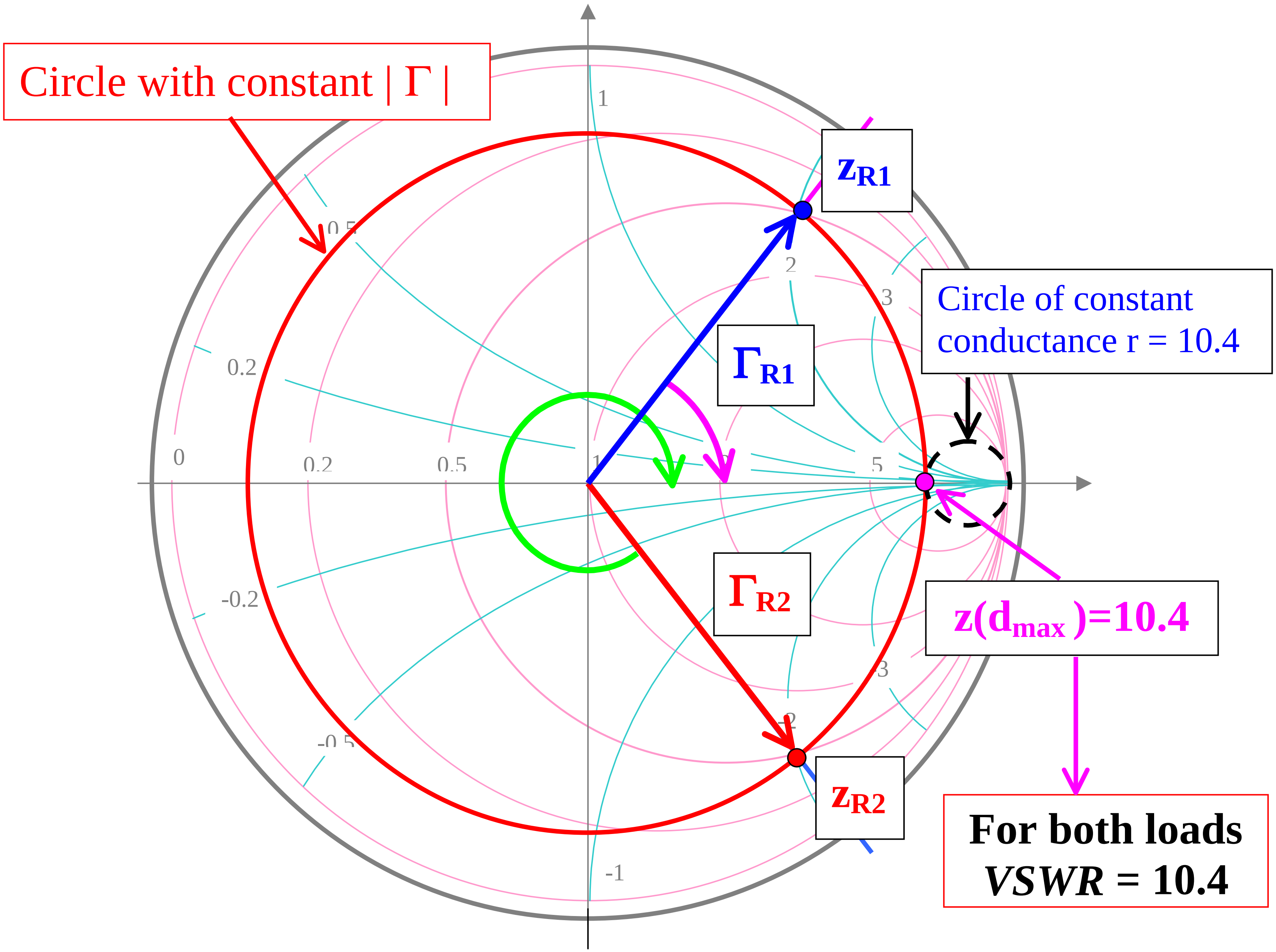
This quantity is always real and ≥ 1. The VSWR is simply obtained on the Smith chart, by reading the value of the (real) normalized impedance, at the location d_{max} where Γ is real and positive.

The graphical step-by-step procedure is:

1. **Identify the load reflection coefficient Γ_R and the normalized load impedance Z_R on the Smith chart.**
2. **Draw the circle of constant reflection coefficient amplitude $|\Gamma(d)| = |\Gamma_R|$.**
3. **Find the intersection of this circle with the real positive axis for the reflection coefficient (corresponding to the transmission line location d_{max}).**
4. **A circle of constant normalized resistance will also intersect this point. Read or interpolate the value of the normalized resistance to determine the *VSWR*.**

Example: Find the *VSWR* for

$$Z_{R1} = 25 + j\,100\ \Omega\ \ ;\ \ Z_{R2} = 25 - j100\Omega \quad (Z_0 = 50\ \Omega)$$


Circle with constant | Γ |
z R1
Γ R1
Γ R2
z R2
Circle of constant conductance r = 10.4
z(d max)=10.4
For both loads
VSWR = 10.4
1
2
3
5
0
0.2
0.5
-0.2
-0.5
-1

Given $Z(\mathrm{d})$ $\Longleftrightarrow$ Find $Y(\mathrm{d})$

Note: The normalized impedance and admittance are defined as

$$z(d) = \frac{1+\Gamma(d)}{1-\Gamma(d)} \qquad\qquad y(d) = \frac{1-\Gamma(d)}{1+\Gamma(d)}$$

Since

$$\Gamma\left(d+\frac{\lambda}{4}\right) = -\Gamma(d)$$

$$\Rightarrow \quad z\left(d+\frac{\lambda}{4}\right) = \frac{1+\Gamma\left(d+\frac{\lambda}{4}\right)}{1-\Gamma\left(d+\frac{\lambda}{4}\right)} = \frac{1-\Gamma(d)}{1+\Gamma(d)} = y(d)$$

Keep in mind that the equality

$$z\left(d+\frac{\lambda}{4}\right)=y(d)$$

is only valid for normalized impedance and admittance. The actual values are given by

$$Z\left(d+\frac{\lambda}{4}\right)=Z_0\cdot z\left(d+\frac{\lambda}{4}\right)$$

$$Y(d)=Y_0\cdot y(d)=\frac{y(d)}{Z_0}$$

where $Y_0 = 1/Z_0$ is the characteristic admittance of the transmission

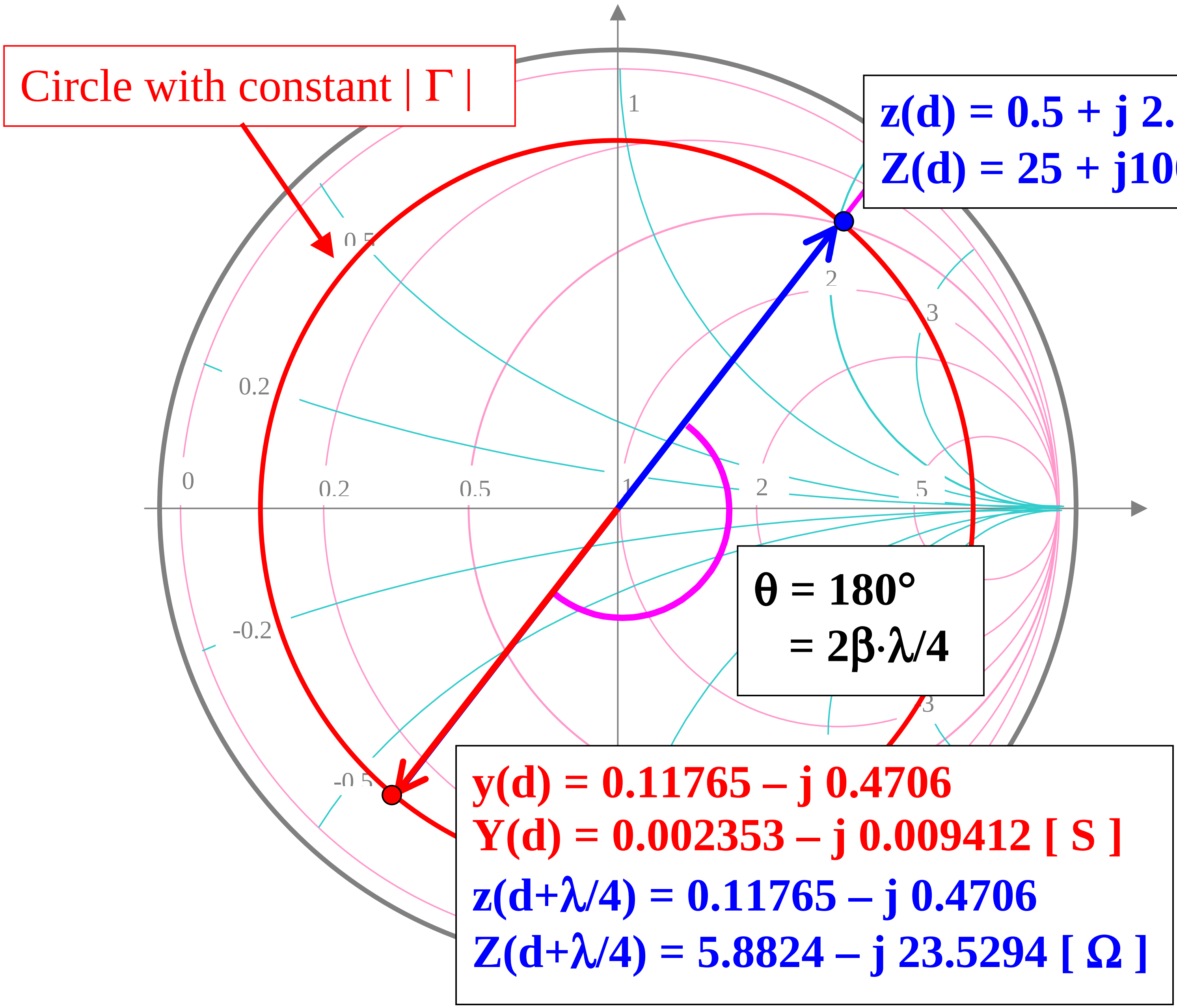
line.
The graphical step-by-step procedure is:

1. **Identify the load reflection coefficient Γ_R and the normalized load impedance z_R on the Smith chart.**
2. **Draw the circle of constant reflection coefficient amplitude $|\Gamma(d)| = |\Gamma_R|$.**
3. **The normalized admittance is located at a point on the circle of constant $|\Gamma|$ which is diametrically opposite to the normalized impedance.**

Example: Given

$$Z_R = 25 + j\,100\ \Omega \qquad \text{with} \qquad Z_0 = 50\ \Omega$$

find Y_R.



Circle with constant | Γ |
z(d) = 0.5 + j 2.0
Z(d) = 25 + j100 [Ω]
θ = 180°
= 2β·λ/4
y(d) = 0.11765 – j 0.4706
Y(d) = 0.002353 – j 0.009412 [S]
z(d+λ/4) = 0.11765 – j 0.4706
Z(d+λ/4) = 5.8824 – j 23.5294 [Ω]

The Smith chart can be used for line admittances, by shifting the space reference to the admittance location. After that, one can move on the chart just reading the numerical values as representing admittances.

Let's review the impedance-admittance terminology:

Impedance = Resistance + j Reactance

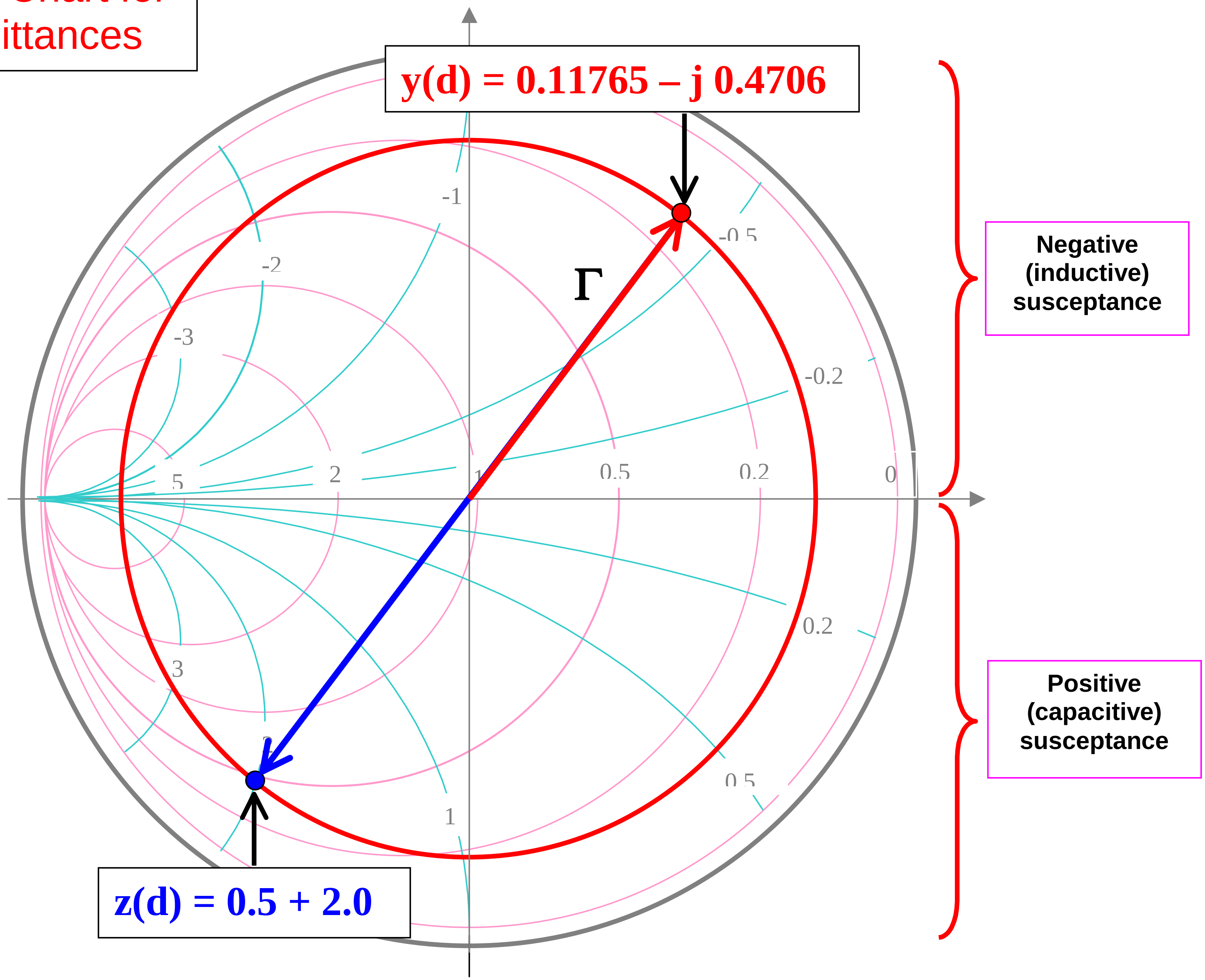
$$Z \quad = \quad R \quad + \quad jX$$

Admittance = Conductance + j Susceptance

$$Y \quad = \quad G \quad + \quad jB$$

On the impedance chart, the correct reflection coefficient is always represented by the vector corresponding to the normalized impedance. Charts specifically prepared for admittances are modified to give the correct reflection coefficient in correspondence of admittance.

Smith Chart for Admittances



Since related impedance and admittance are on opposite sides of the same Smith chart, the imaginary parts always have different sign.

Therefore, a positive (inductive) reactance corresponds to a negative (inductive) susceptance, while a negative (capacitive) reactance corresponds to a positive (capacitive) susceptance.

Numerically, we have

$$z = r + j\,x \qquad\qquad y = g + j\,b = \frac{1}{r + j\,x}$$

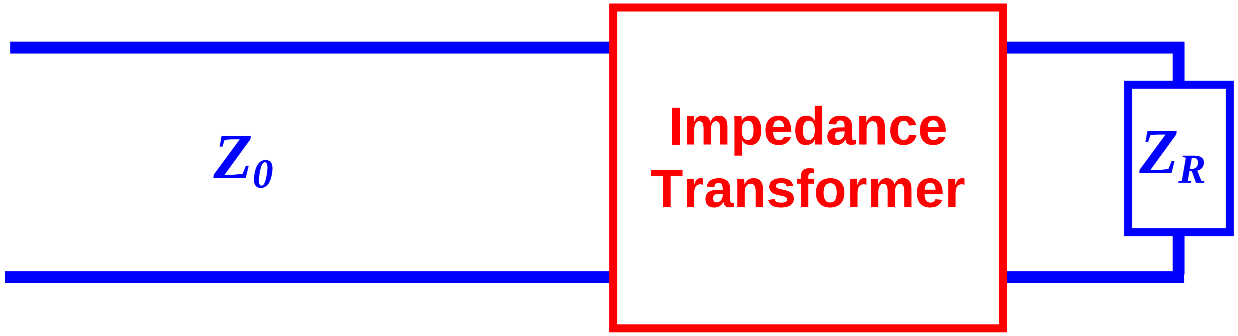
$$y = \frac{r - j\,x}{(r + j\,x)(r - j\,x)} = \frac{r - jx}{r^2 + x^2}$$

$$\Rightarrow \qquad g = \frac{r}{r^2 + x^2} \qquad\qquad b = -\frac{x}{r^2 + x^2}$$

Impedance Matching

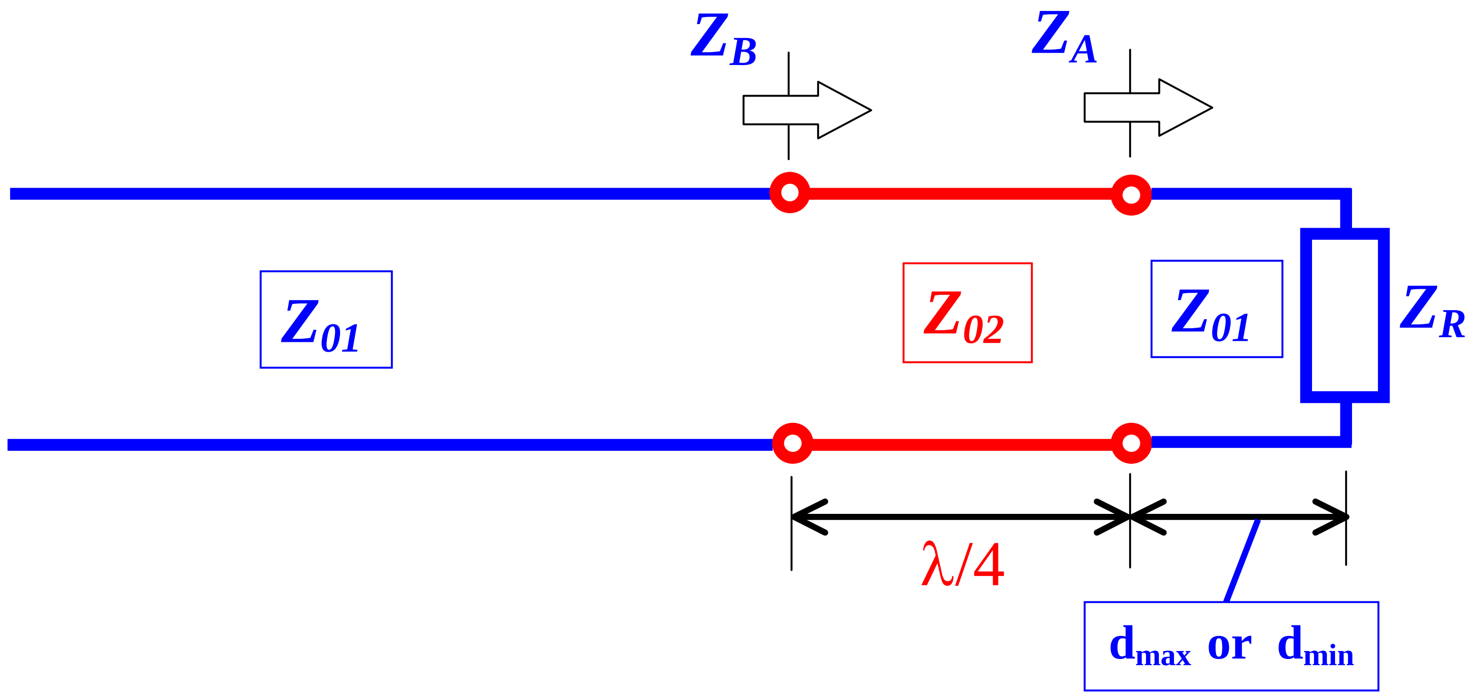
A number of techniques can be used to eliminate reflections when line characteristic impedance and load impedance are mismatched. Impedance matching techniques can be designed to be effective for a specific frequency of operation (narrow band techniques) or for a given frequency spectrum (broadband techniques).

One method of impedance matching involves the insertion of an impedance transformer between line and load



In the following, we neglect effects of loss in the lines.

A simple narrow band impedance transformer consists of a transmission line section of length $\lambda /4$



The impedance transformer is positioned so that it is connected to a real impedance Z_A. This is always possible if a location of maximum or minimum voltage standing wave pattern is selected.

Consider a general load impedance with its corresponding load reflection coefficient

$$Z_R = R_R + jX_R \quad ; \qquad \Gamma_R = \frac{Z_R - Z_{01}}{Z_R + Z_{01}} = |\Gamma_R| \exp(j\phi)$$

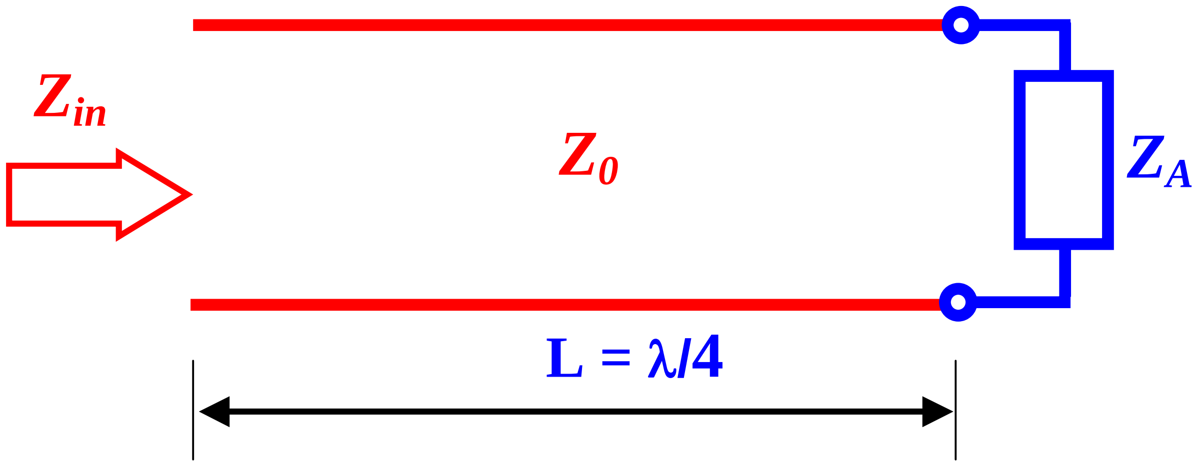
If the transformer is inserted at a location of voltage maximum d_{max}

$$Z_A = Z_{01} \frac{1+\Gamma(d)}{1-\Gamma(d)} = Z_{01} \frac{1+|\Gamma_R|}{1-|\Gamma_R|}$$

If it is inserted instead at a location of voltage minimum d_{min}

$$Z_A = Z_{01} \frac{1+\Gamma(d)}{1-\Gamma(d)} = Z_{01} \frac{1-|\Gamma_R|}{1+|\Gamma_R|}$$

Consider now the input impedance of a line of length $\lambda/4$



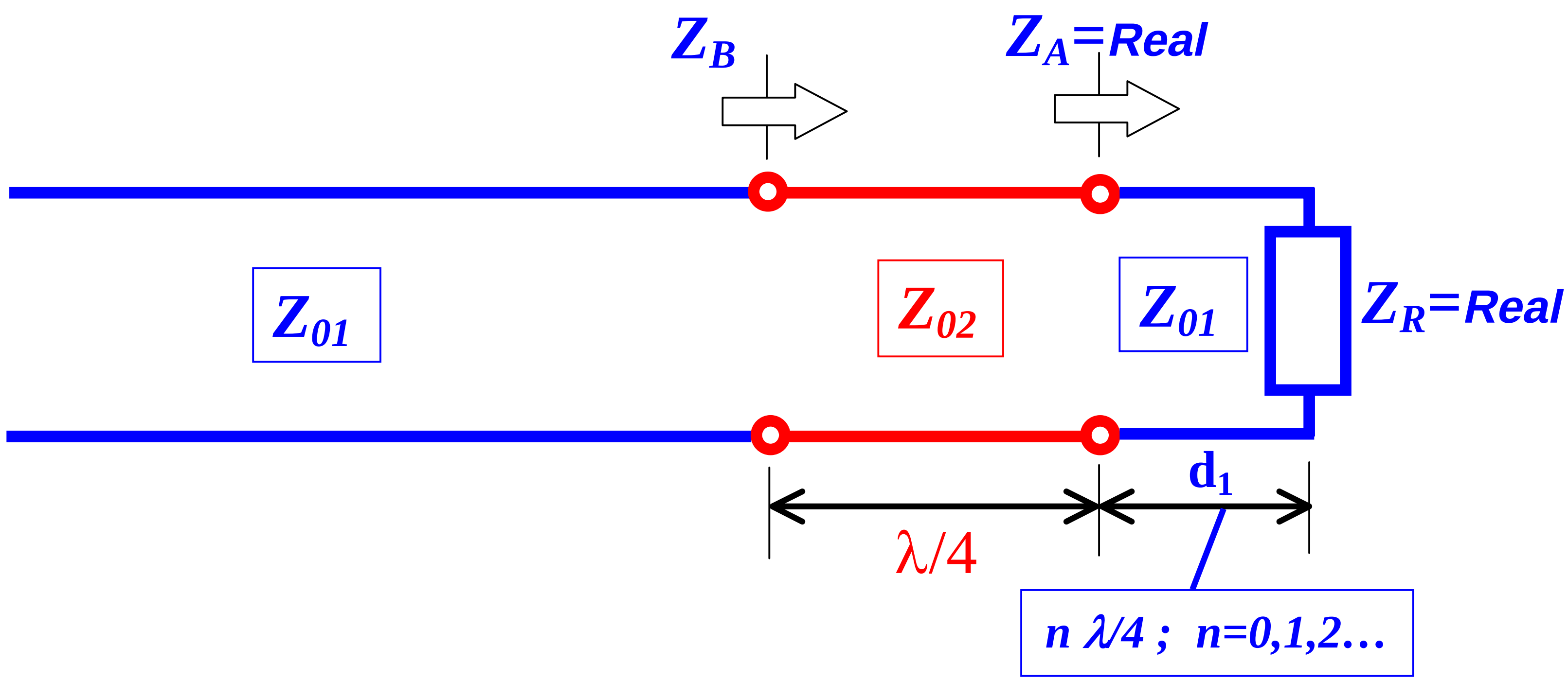
Since:

$$Z_A = Z_{01}\frac{1+\Gamma(d)}{1-\Gamma(d)} = Z_{01}\frac{1-|\Gamma_R|}{1+|\Gamma_R|}$$

we have

$$Z_{in} = \lim_{\tan(\beta L)\to\infty} Z_0 \frac{Z_A + jZ_0\tan(\beta L)}{jZ_A\tan(\beta L) + Z_0} \rightarrow \frac{Z_0^2}{Z_A}$$

Note that if the load is real, the voltage standing wave pattern at the load is maximum when $Z_R > Z_{01}$ or minimum when $Z_R < Z_{01}$. The transformer can be connected directly at the load location or at a distance from the load corresponding to a multiple of $\lambda/4$.



If the load impedance is real and the transformer is inserted at a distance from the load equal to an even multiple of λ/4 then

$$Z_A = Z_R \quad ; \qquad d_1 = 2n\frac{\lambda}{4} = n\frac{\lambda}{2}$$

but if the distance from the load is an odd multiple of λ/4

$$Z_A = \frac{Z_{01}^2}{Z_R} \quad ; \qquad d_1 = (2n+1)\frac{\lambda}{4} = n\frac{\lambda}{2} + \frac{\lambda}{4}$$

The input impedance of the impedance transformer after inclusion in the circuit is given by

$$Z_B = \frac{Z_{02}^2}{Z_A}$$

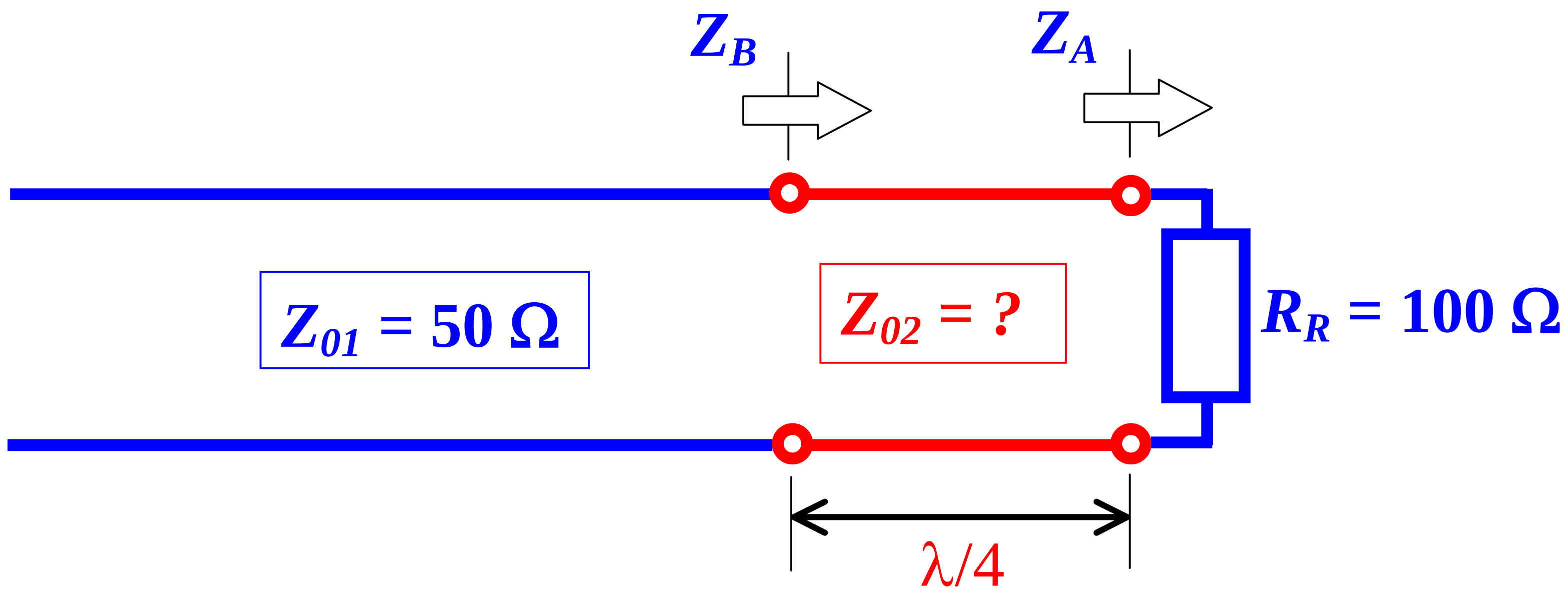
For impedance matching we need

$$Z_{01} = \frac{Z_{02}^2}{Z_A} \qquad \Rightarrow \qquad Z_{02} = \sqrt{Z_{01} Z_A}$$

The characteristic impedance of the transformer is simply the geometric average between the characteristic impedance of the original line and the load seen by the transformer.

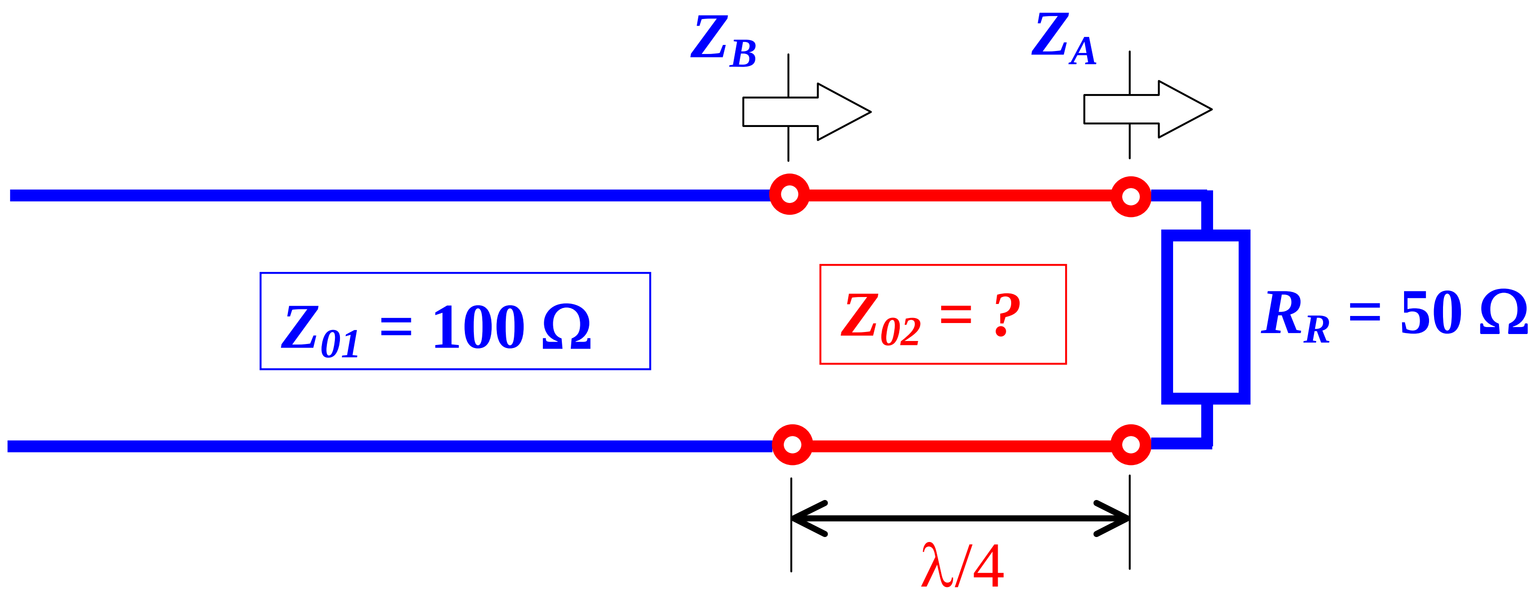
Let's now review some simple examples.

❑ Real Load Impedance



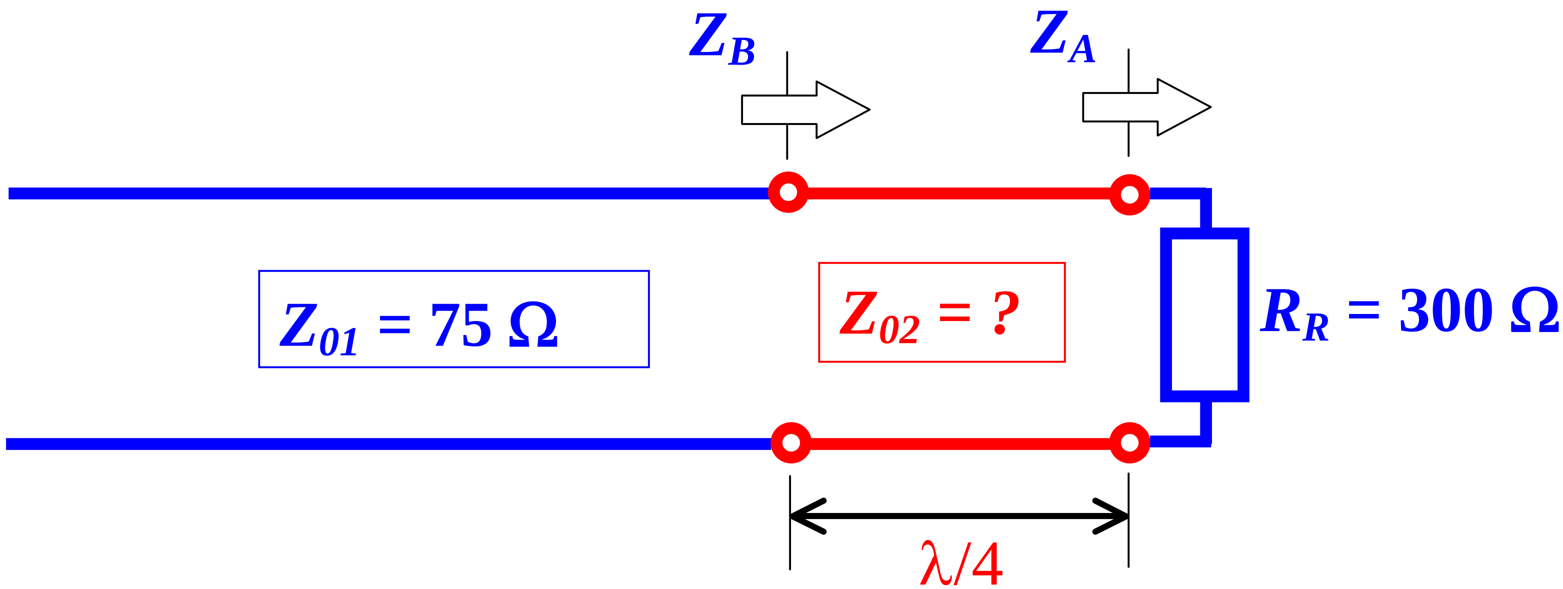
$$Z_B = \frac{Z_{02}^2}{R_R} = Z_{01} \Rightarrow \; Z_{02} = \sqrt{Z_{01} R_R} = \sqrt{50 \cdot 100} \approx 70.71\,\Omega$$

Note that an identical result is obtained by switching Z_{01} and R_R



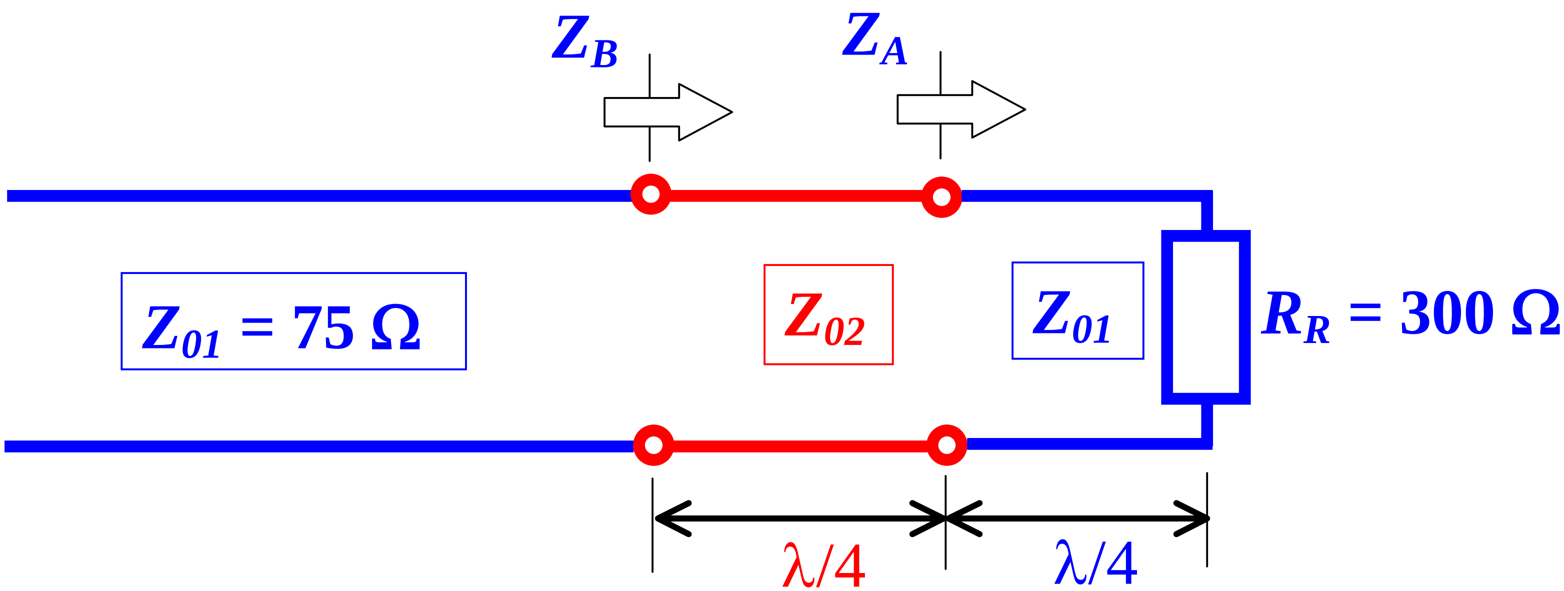
$$Z_B = \frac{Z_{02}^2}{R_R} = Z_{01} \Rightarrow \; Z_{02} = \sqrt{Z_{01} R_R} = \sqrt{100 \cdot 50} \approx 70.71\,\Omega$$

Another real load case



$$Z_B = \frac{Z_{02}^2}{R_R} = Z_{01} \Rightarrow Z_{02} = \sqrt{Z_{01} R_R} = \sqrt{75 \cdot 300} = 150\ \Omega$$

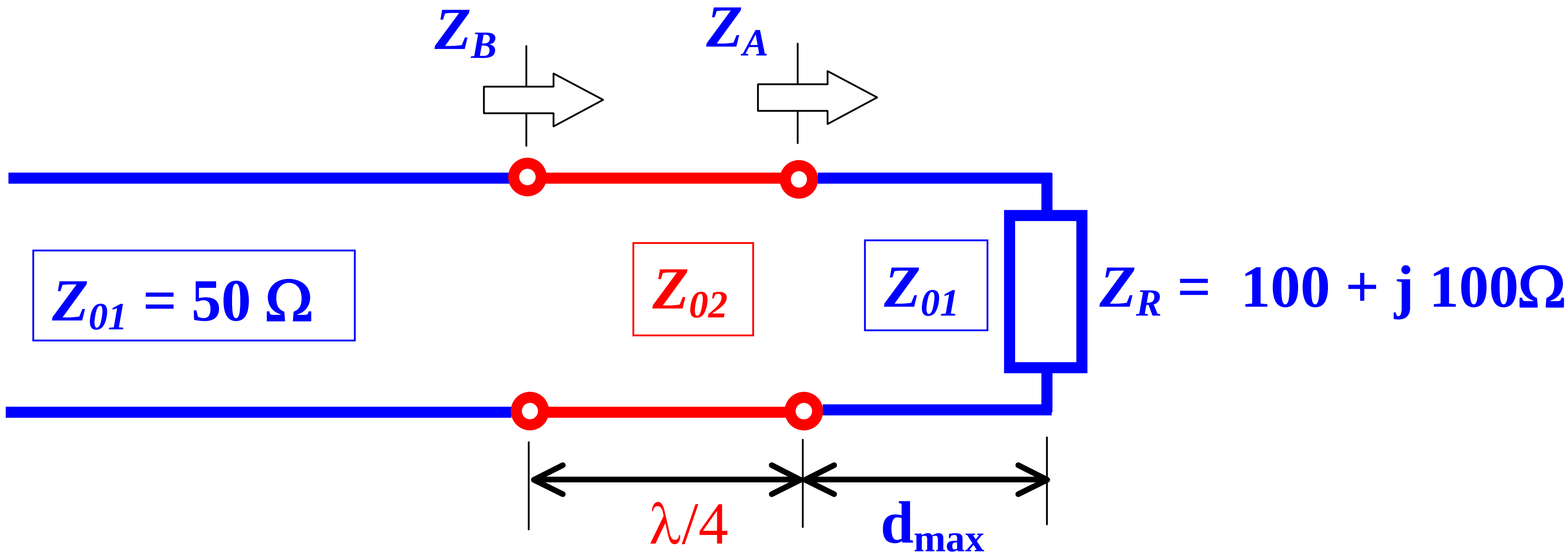
Same impedances as before, but now the transformer is inserted at a distance λ/4 from the load (voltage minimum in this case)



$$Z_A = \frac{Z_{01}^2}{R_R} = \frac{75^2}{300} = 18.75\ \Omega$$

$$Z_B = \frac{Z_{02}^2}{Z_A} = Z_{01} \Rightarrow\ Z_{02} = \sqrt{Z_{01} Z_A} = \sqrt{75 \cdot 18.75} = 37.5\ \Omega$$

❑ Complex Load Impedance – Transformer at voltage maximum

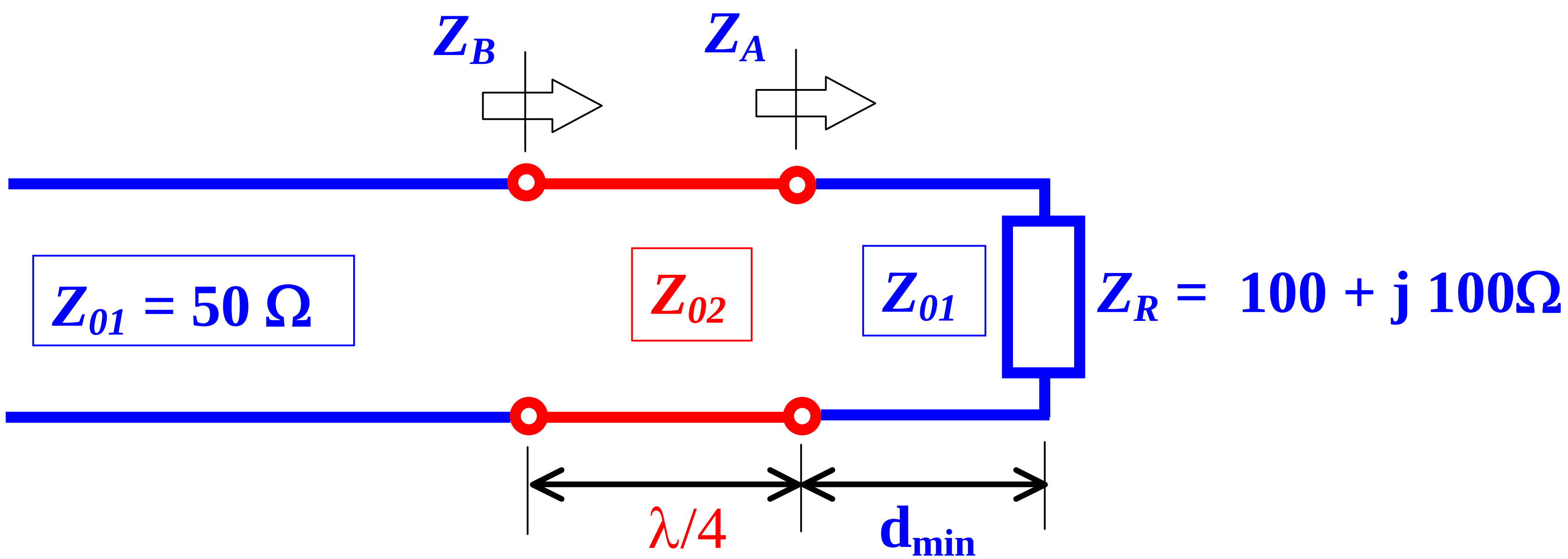


$$\left|\Gamma_R\right| = \left|\frac{100 + j100 - 50}{100 + j100 + 50}\right| \approx 0.62$$

$$Z_A = Z_0 \frac{1 + \left|\Gamma_R\right|}{1 - \left|\Gamma_R\right|} \approx 213.28\Omega$$

$$Z_{02} = \sqrt{Z_{01} Z_A} = \sqrt{50 \cdot 213.28} = 103.27\ \Omega$$

❑ Complex Load Impedance – Transformer at voltage minimum

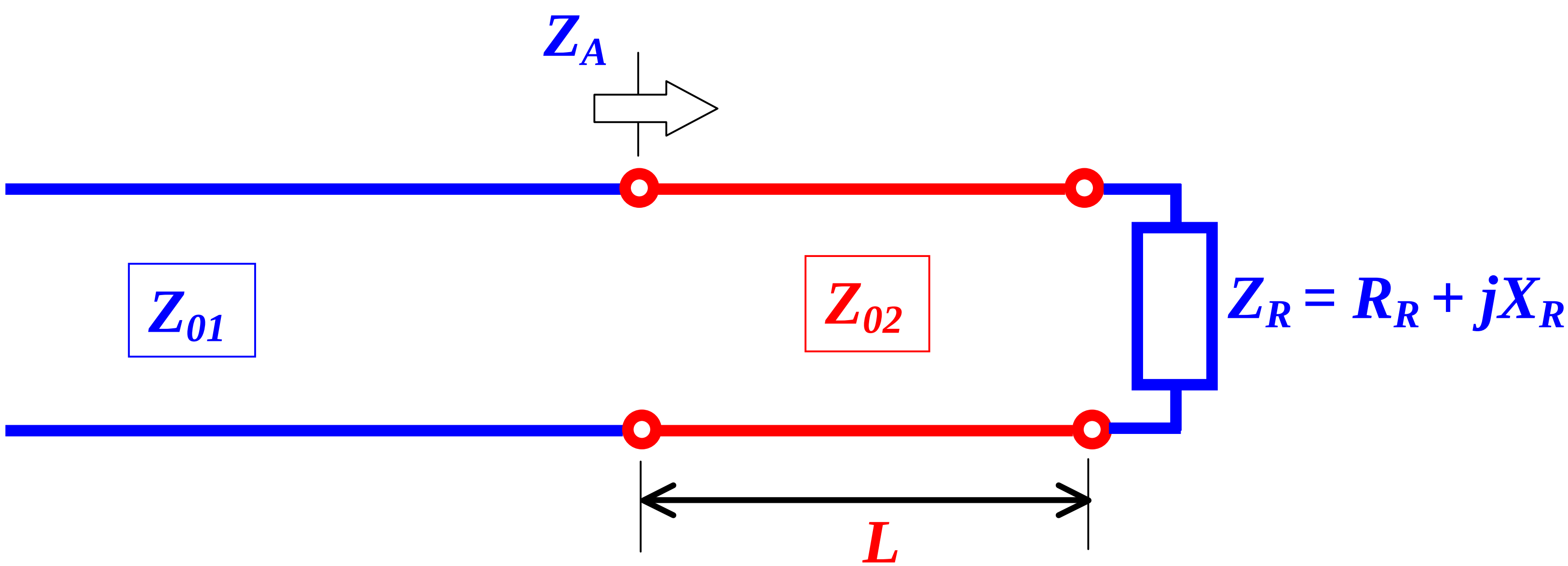


$$|\Gamma_R| = \left|\frac{100 + j100 - 50}{100 + j100 + 50}\right| \approx 0.62$$

$$Z_A = Z_0 \frac{1 - |\Gamma_R|}{1 + |\Gamma_R|} \approx 11.72\ \Omega$$

$$Z_{02} = \sqrt{Z_{01} Z_A} = \sqrt{50 \cdot 11.72} = 24.21\ \Omega$$

If it is not important to realize the impedance transformer with a quarter wavelength line, we can try to select a transmission line with appropriate length and characteristic impedance, such that the input impedance is the required real value



$$Z_{01} = Z_A = Z_{02} \frac{R_R + jX_R + jZ_{02} \tan(\beta L)}{Z_{02} + j\left(R_R + jX_R\right) \tan(\beta L)}$$

After separation of real and imaginary parts we obtain the equations

$$Z_{02}(Z_{01} - R_R) = Z_{01} X_R \tan(\beta L)$$

$$\tan(\beta L) = \frac{Z_{02} X_R}{Z_{01} R_R - Z_{02}^2}$$

with final solution

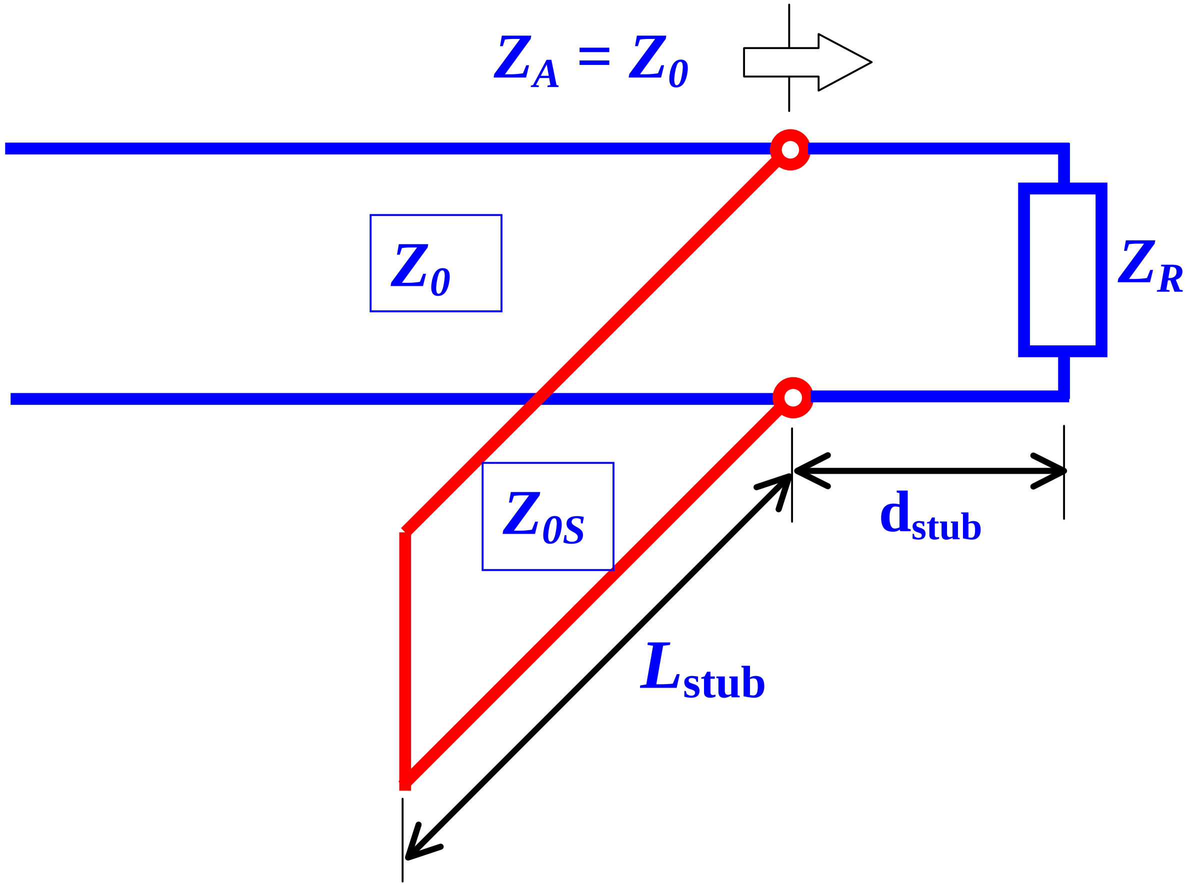
$$Z_{02} = \frac{\sqrt{Z_{01} R_R - R_R^2 - X_R^2}}{\sqrt{1 - R_R / Z_{01}}}$$

$$\tan(\beta L) = \frac{\sqrt{(1 - R_R / Z_{01})\left(Z_{01} R_R - R_R^2 - X_R^2\right)}}{X_R}$$

The transformer can be realized as long as the result for Z_{02} is real. Note that this is also a narrow band approach.

❑ Single stub impedance matching

Impedance matching can be achieved by inserting another transmission line (stub) as shown in the diagram below



There are two design parameters for single stub matching:

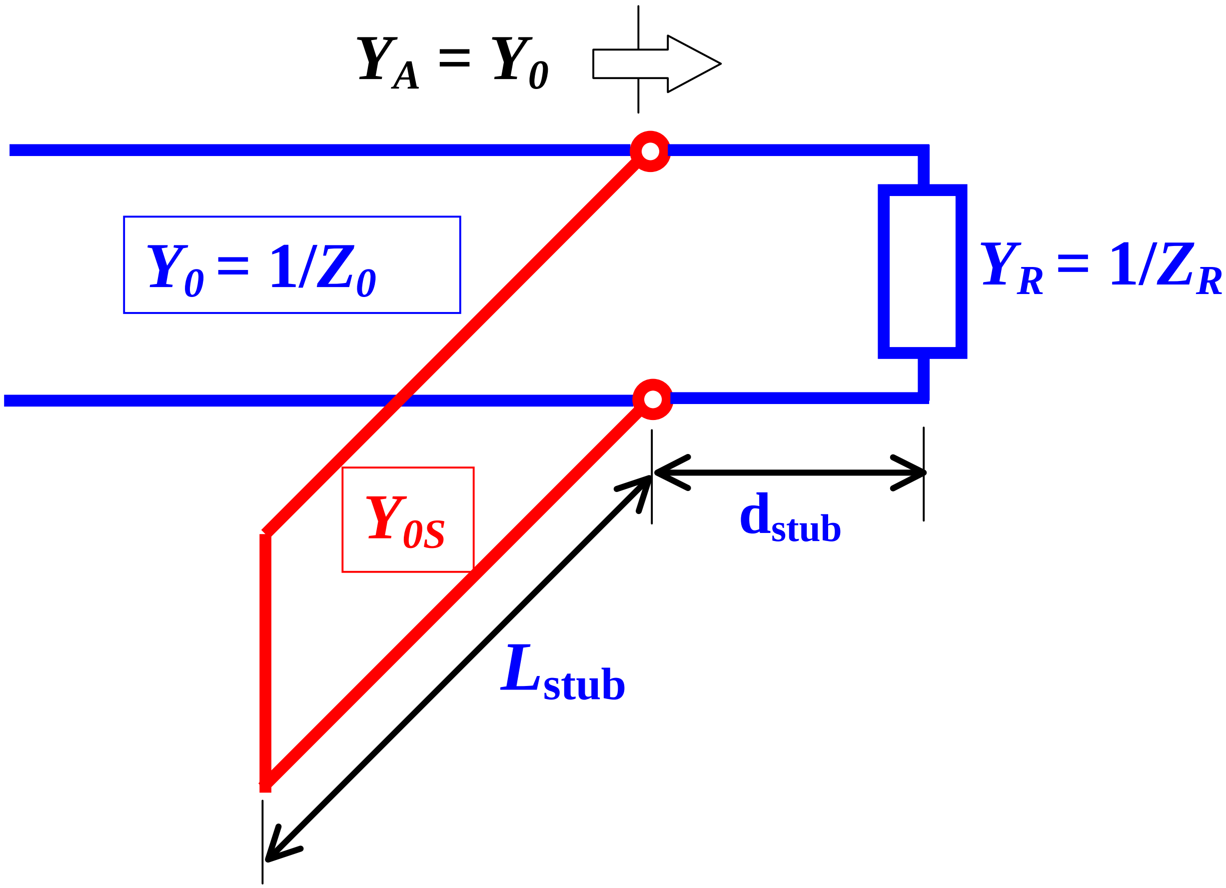
- **The location of the stub with reference to the load $\mathbf{d}_{\text{stub}}$**

- **The length of the stub line L_{stub}**

Any load impedance can be matched to the line by using single stub technique. The drawback of this approach is that if the load is changed, the location of insertion may have to be moved.

The transmission line realizing the stub is normally terminated by a short or by an open circuit. In many cases it is also convenient to select the same characteristic impedance used for the main line, although this is not necessary. The choice of open or shorted stub may depend in practice on a number of factors. A short circuited stub is less prone to leakage of electromagnetic radiation and is somewhat easier to realize. On the other hand, an open circuited stub may be more practical for certain types of transmission lines, for example microstrips where one would have to drill the insulating substrate to short circuit the two conductors of the line.

Since the circuit is based on insertion of a parallel stub, it is more convenient to work with admittances, rather than impedances.

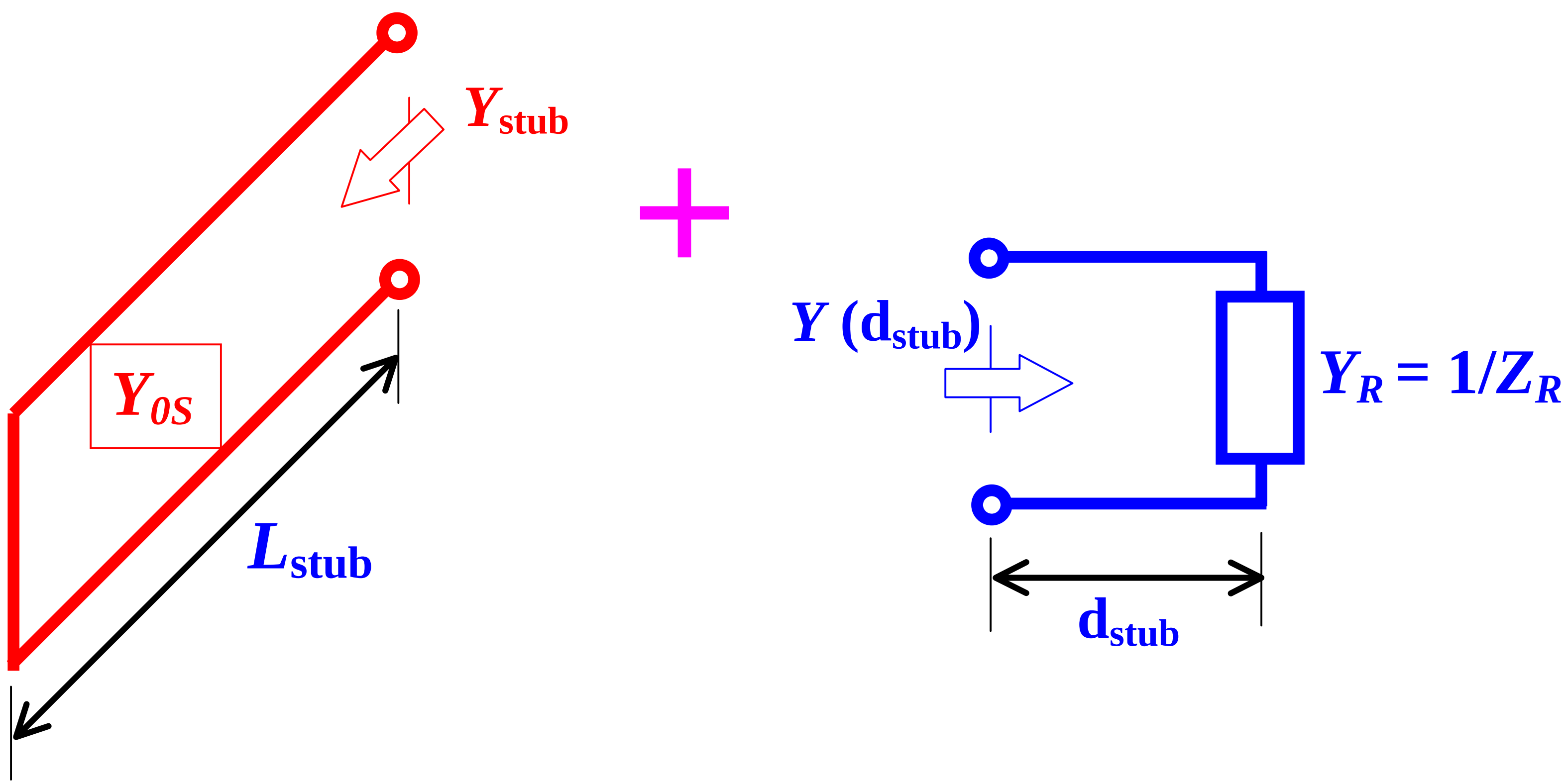


For proper impedance match:

$$Y_A = Y_{stub} + Y(d_{stub}) = Y_0 = \frac{1}{Z_0}$$

Input admittance of the stub line

Line admittance at location d_{stub} before the stub is applied



In order to complete the design, we have to find an appropriate location for the stub. Note that the input admittance of a stub is always imaginary (inductance if negative, or capacitance if positive)

$$Y_{\text{stub}} = jB_{\text{stub}}$$

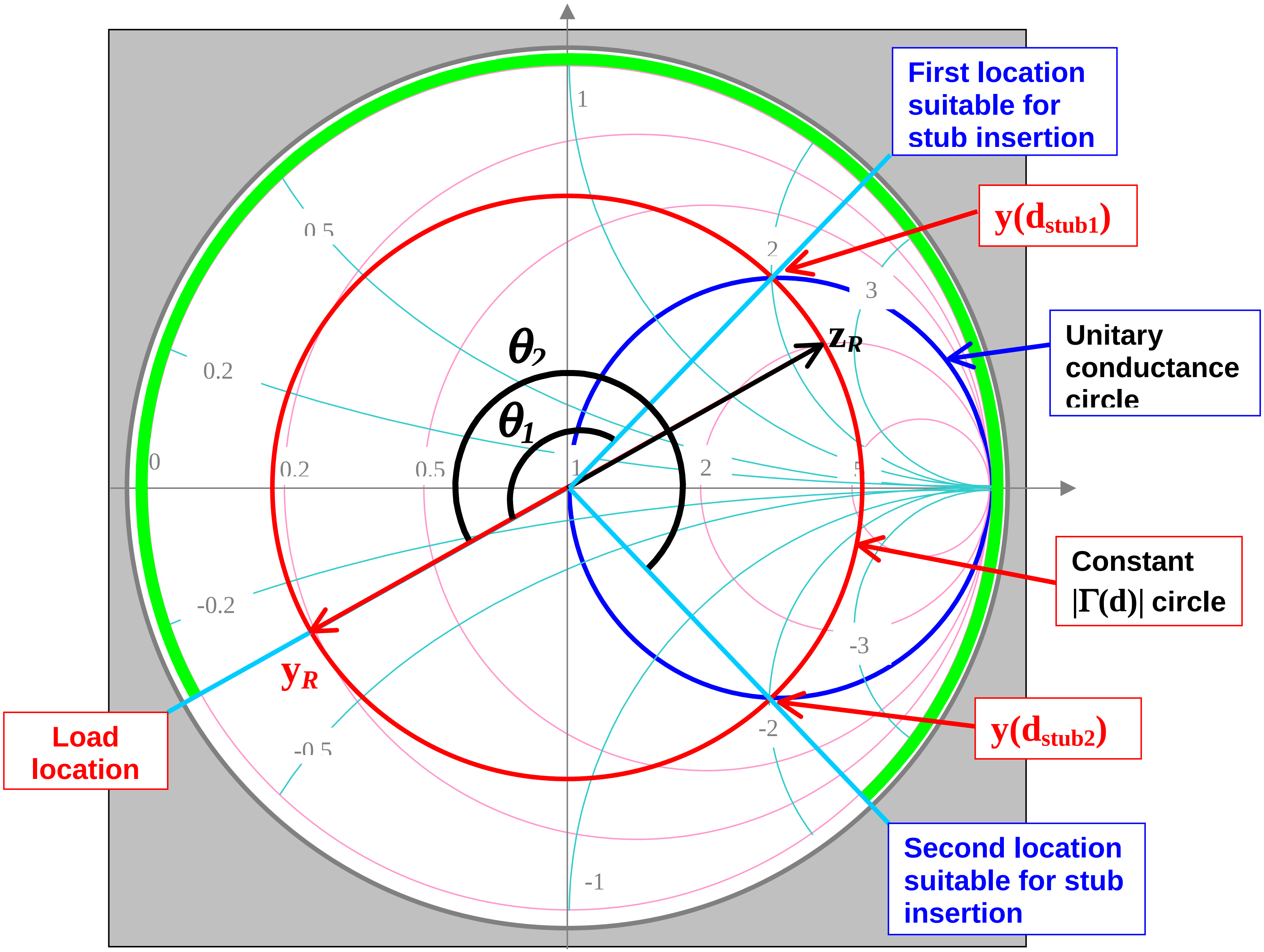
A stub should be placed at a location where the line admittance has real part equal to Y_0

$$Y(\mathrm{d}_{\text{stub}}) = Y_0 + jB(\mathrm{d}_{\text{stub}})$$

For matching, we need to have

$$B_{\text{stub}} = -B(\mathrm{d}_{\text{stub}})$$

Depending on the length of the transmission line, there may be a number of possible locations where a stub can be inserted for impedance matching. It is very convenient to analyze the possible solutions on a Smith chart.


First location suitable for stub insertion
y(d_stub1)
Unitary conductance circle
Constant |Γ(d)| circle
y(d_stub2)
Second location suitable for stub insertion
Load location
θ2
θ1
zR
yR
1
0.5
0.2
0
0.2
0.5
1
2
3
5
2
-0.2
-0.5
-1
-2
-3

The red arrow on the example indicates the load admittance. This provides on the "admittance chart" the physical reference for the load location on the transmission line. Notice that in this case the load admittance falls outside the unitary conductance circle. If one moves from load to generator on the line, the corresponding chart location moves from the reference point, in clockwise motion, according to an angle θ (indicated by the light green arc)

$$\theta = 2\beta d = \frac{4\pi}{\lambda} d$$

The value of the admittance rides on the red circle which corresponds to constant magnitude of the line reflection coefficient, $|\Gamma(d)| = |\Gamma_R|$, imposed by the load.

Every circle of constant $|\Gamma(d)|$ intersects the circle $\text{Re}\{ y \} = 1$ (unitary normalized conductance), in correspondence of two points. Within the first revolution, the two intersections provide the locations closest to the load for possible stub insertion.

The first solution corresponds to an admittance value with positive imaginary part, in the upper portion of the chart

Line Admittance - Actual: $Y(d_{stub_1}) = Y_0 + j\,B(d_{stub_1})$

Normalized: $y(d_{stub_1}) = 1 + j\,b(d_{stub_1})$

Stub Location: $d_{stub_1} = \dfrac{\theta_1}{4\pi}\lambda$

Stub Admittance - Actual: $-j\,B(d_{stub_1})$

Normalized: $-j\,b(d_{stub_1})$

Stub Length: $L_{stub} = \dfrac{\lambda}{2\pi}\tan^{-1}\left(\dfrac{1}{Z_{0s}B(d_{stub_1})}\right)$ **(short)**

$L_{stub} = \dfrac{\lambda}{2\pi}\tan^{-1}\left(Z_{0s}B(d_{stub_1})\right)$ **(open)**

The second solution corresponds to an admittance value with negative imaginary part, in the lower portion of the chart

Line Admittance - Actual: $Y(\mathrm{d}_{stub_2}) = Y_0 - j\,B(\mathrm{d}_{stub_2})$

Normalized: $y(\mathrm{d}_{stub_2}) = 1 - j\,b(\mathrm{d}_{stub_2})$

Stub Location: $\mathrm{d}_{stub_2} = \dfrac{\theta_2}{4\pi}\lambda$

Stub Admittance - Actual: $j\,B(\mathrm{d}_{stub_2})$

Normalized: $j\,b(\mathrm{d}_{stub_2})$

Stub Length: $L_{stub} = \dfrac{\lambda}{2\pi}\tan^{-1}\left(-\dfrac{1}{Z_{0s}B(\mathrm{d}_{stub_2})}\right)$ **(short)**

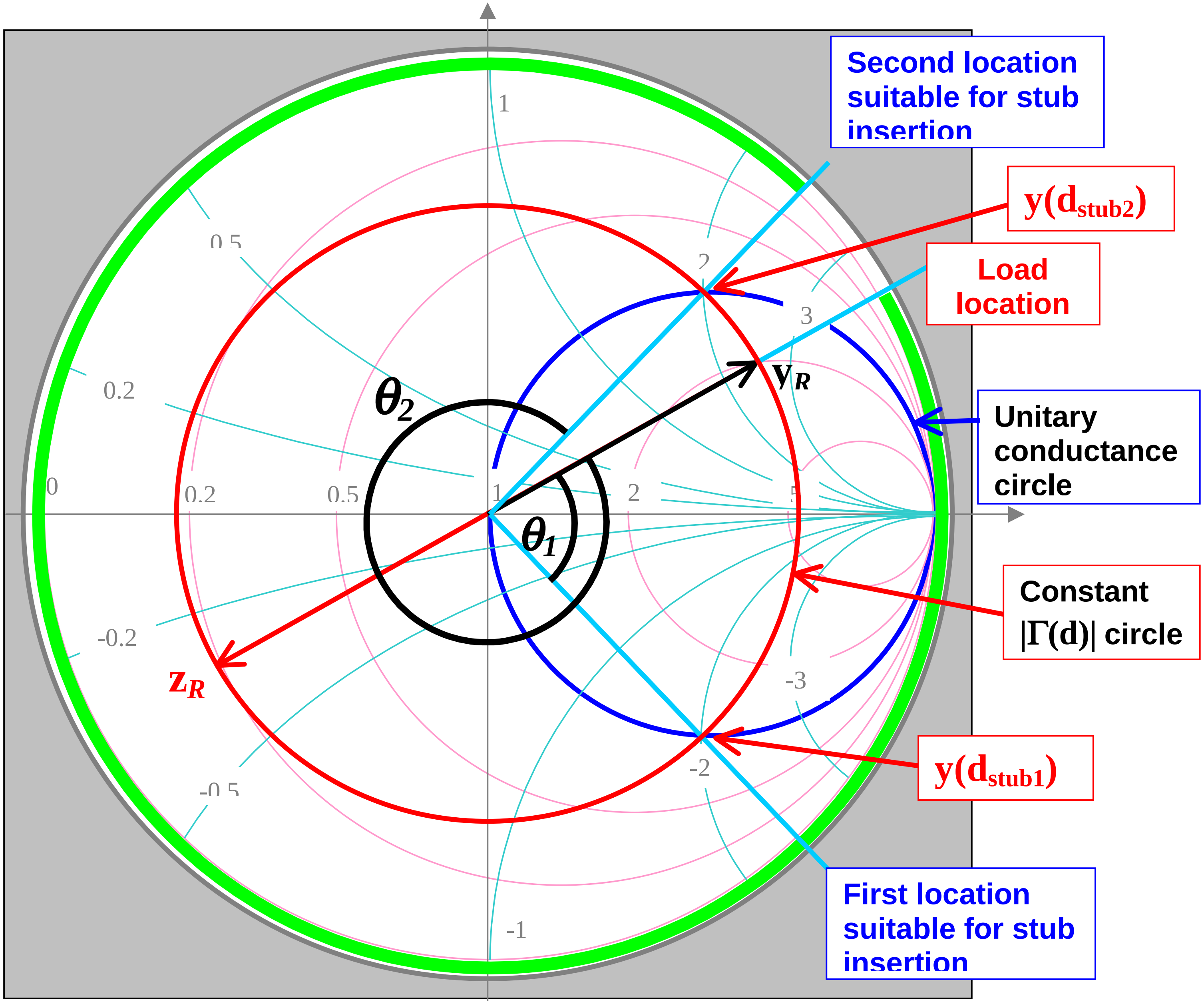
$L_{stub} = \dfrac{\lambda}{2\pi}\tan^{-1}\left(-Z_{0s}B(\mathrm{d}_{stub_2})\right)$ **(open)**

If the normalized load admittance falls inside the unitary conductance circle (see next figure), the first possible stub location corresponds to a line admittance with negative imaginary part. The second possible location has line admittance with positive imaginary part. In this case, the formulae given above for first and second solution exchange place.

If one moves further away from the load, other suitable locations for stub insertion are found by moving toward the generator, at distances multiple of half a wavelength from the original solutions. These locations correspond to the same points on the Smith chart.

$$\text{First set of locations} = d_{stub_1} + n\frac{\lambda}{2}$$

$$\text{Second set of locations} = d_{stub_2} + n\frac{\lambda}{2}$$

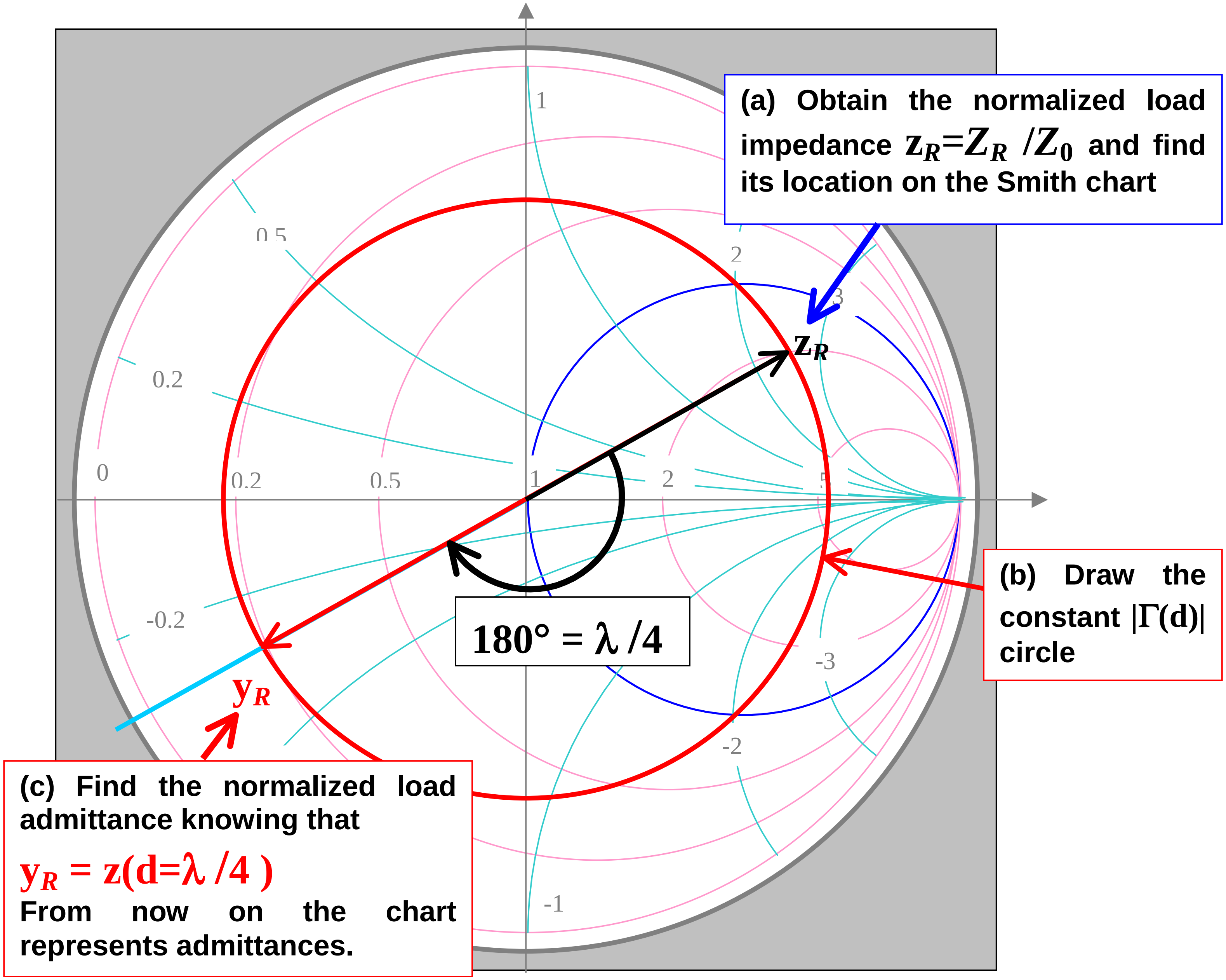

Second location suitable for stub insertion
y(d_stub2)
Load location
Unitary conductance circle
Constant |Γ(d)| circle
y(d_stub1)
First location suitable for stub insertion
θ2
θ1
yR
zR
1
0.5
0.2
0
0.2
0.5
1
2
5
2
3
-0.2
-0.5
-1
-2
-3

Single stub matching problems can be solved on the Smith chart graphically, using a compass and a ruler. This is a step-by-step summary of the procedure:

(a) Find the normalized load impedance and determine the corresponding location on the chart.

(b) Draw the circle of constant magnitude of the reflection coefficient $|\Gamma|$ for the given load.

(c) Determine the normalized load admittance on the chart. This is obtained by rotating 180° on the constant $|\Gamma|$ circle, from the load impedance point. From now on, all values read on the chart are normalized admittances.



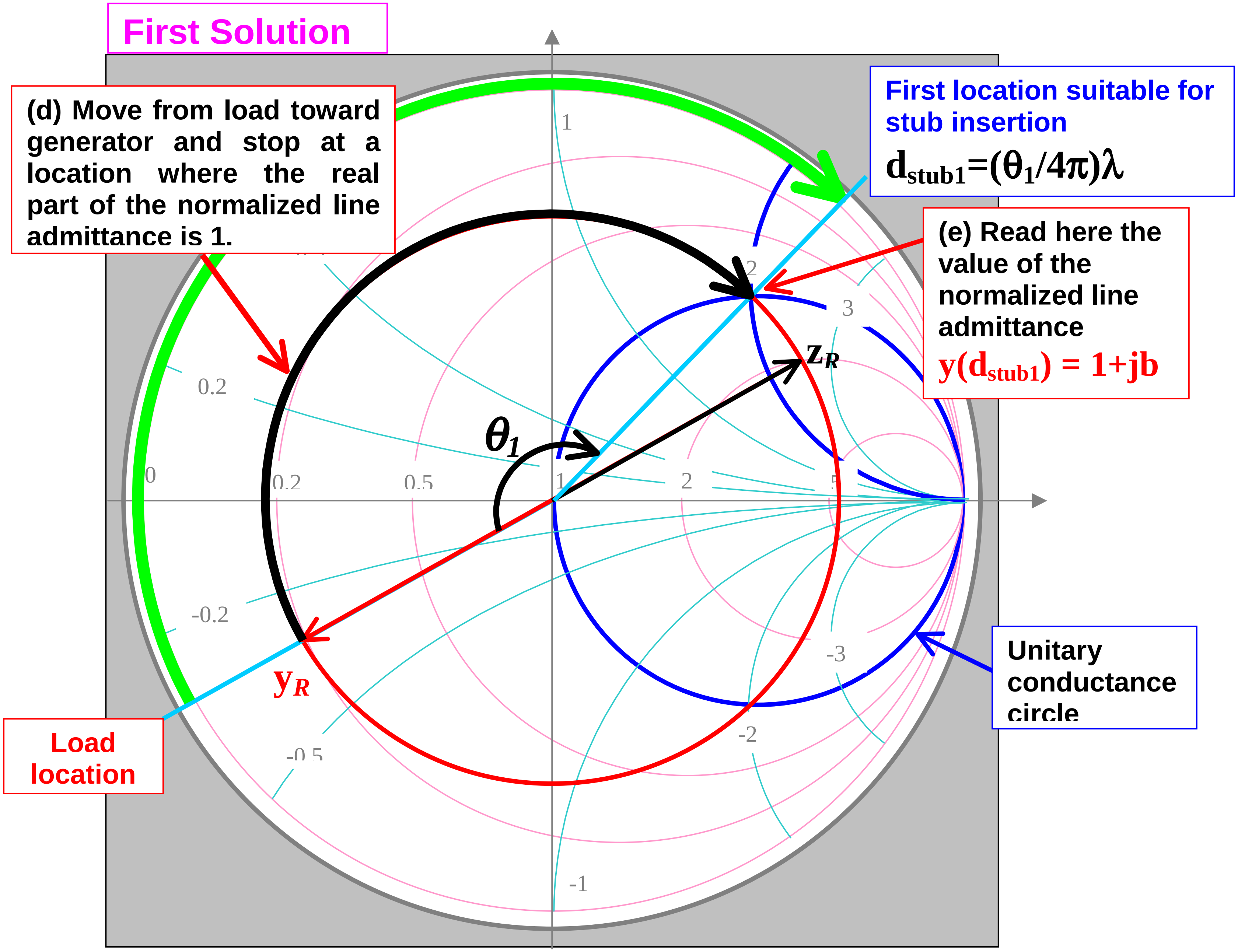
1
0.5
0.2
0
0.2
0.5
1
2
2
3
5
-0.2
-3
-2
-1
z_R
y_R
180° = λ /4
(a) Obtain the normalized load impedance $z_R = Z_R / Z_0$ and find its location on the Smith chart
(b) Draw the constant $|\Gamma(d)|$ circle
(c) Find the normalized load admittance knowing that
$y_R = z(d = \lambda/4)$
From now on the chart represents admittances.

(d) Move from load admittance toward generator by riding on the constant $|\Gamma|$ circle, until the intersections with the unitary normalized conductance circle are found. These intersections correspond to possible locations for stub insertion. Commercial Smith charts provide graduations to determine the angles of rotation as well as the distances from the load in units of wavelength.

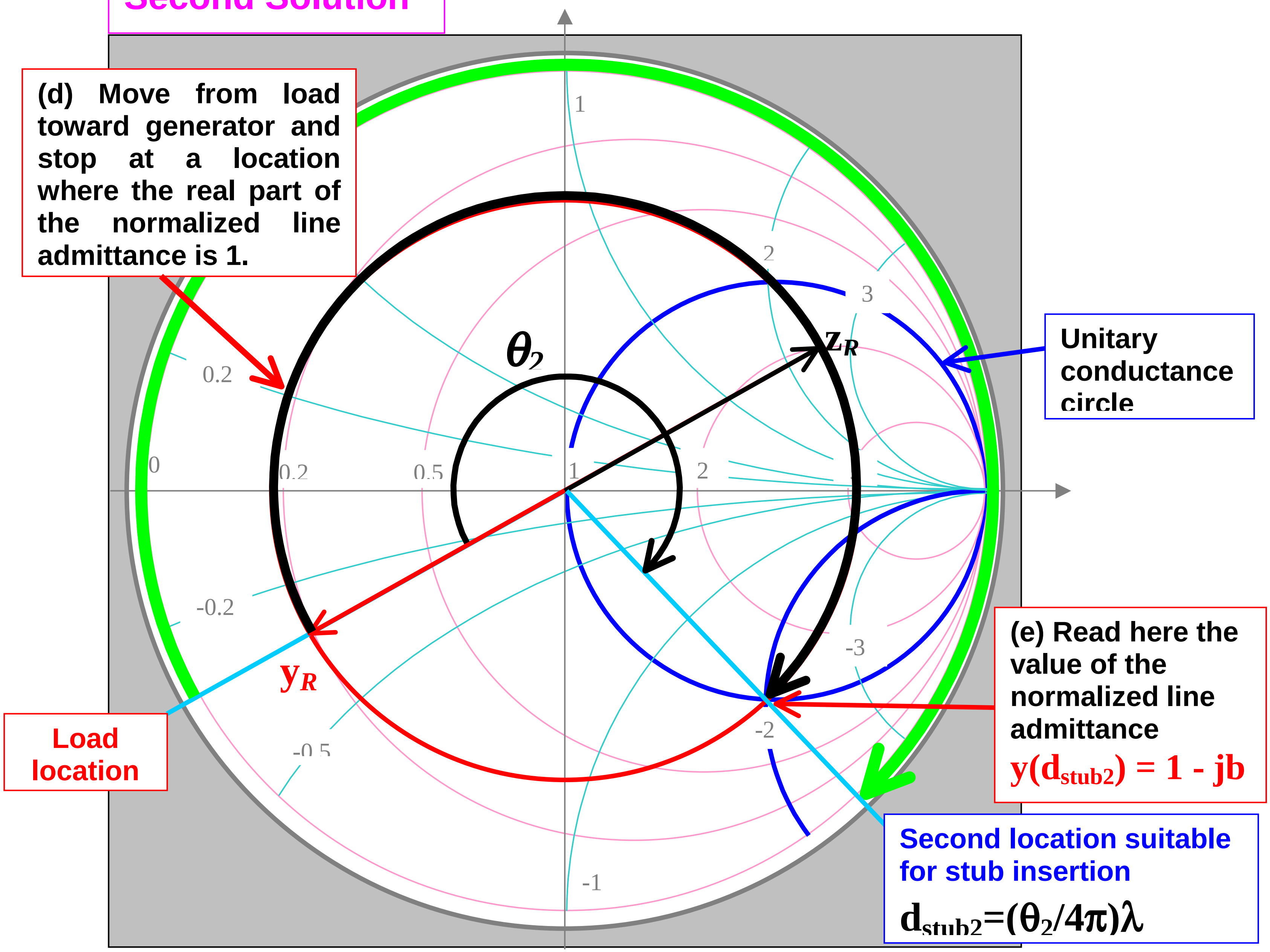
(e) Read the line normalized admittance in correspondence of the stub insertion locations determined in (d). These values will always be of the form

$$y(\mathrm{d_{stub}}) = 1 + jb \qquad \text{top half of chart}$$

$$y(\mathrm{d_{stub}}) = 1 - jb \qquad \text{bottom half of chart}$$


First Solution
(d) Move from load toward generator and stop at a location where the real part of the normalized line admittance is 1.
First location suitable for stub insertion
$d_{stub1}=(\theta_1/4\pi)\lambda$
(e) Read here the value of the normalized line admittance
$y(d_{stub1}) = 1+jb$
z_R
θ_1
y_R
Load location
Unitary conductance circle
1
0.2
0
0.2
0.5
1
2
3
-0.2
-0.5
-1
-2
-3
2

Second Solution



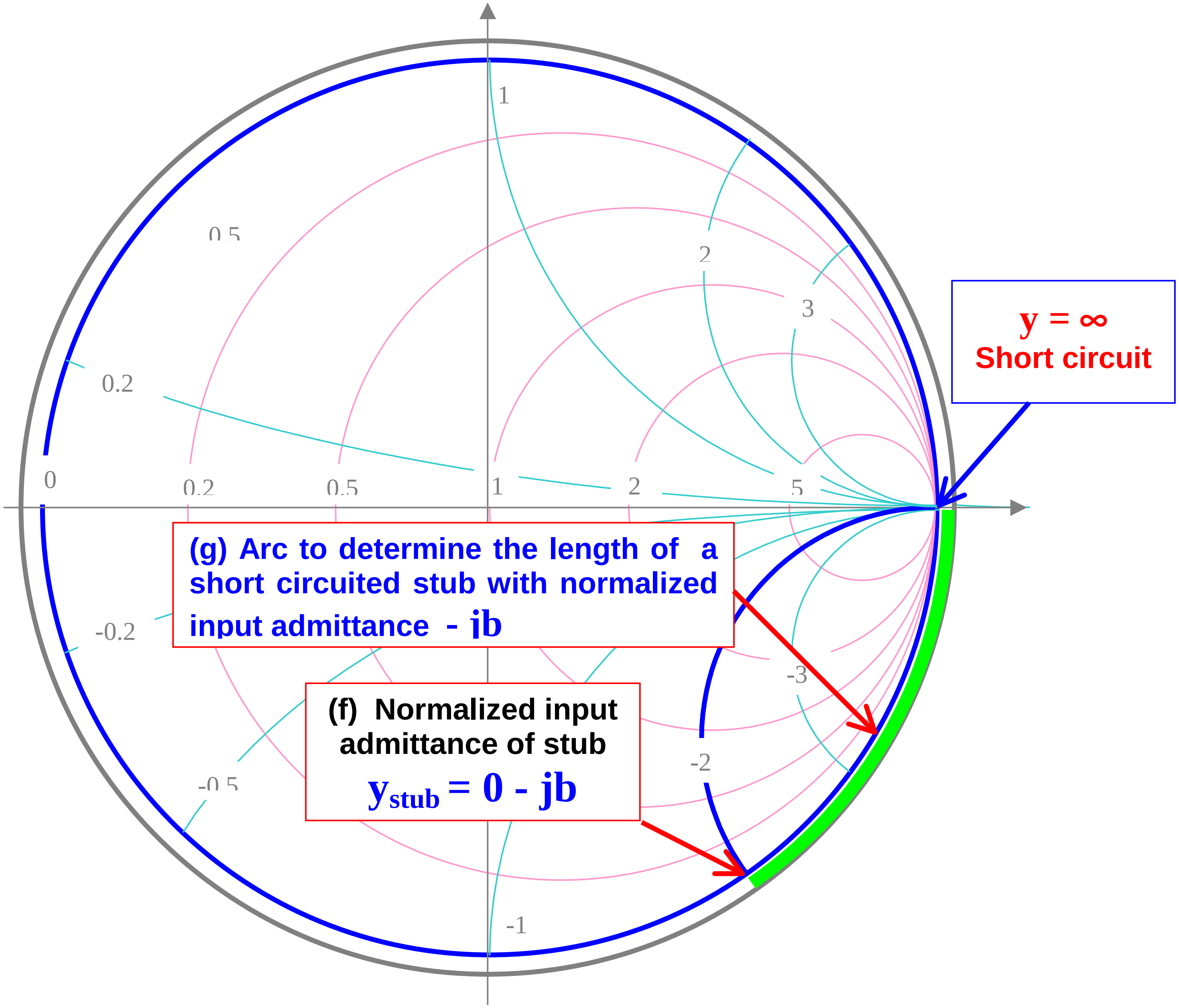
(f) Select the input normalized admittance of the stubs, by taking the opposite of the corresponding imaginary part of the line admittance

$$\text{line: } y(\mathrm{d_{stub}}) = 1 + jb \quad \rightarrow \quad \text{stub: } y_{\text{stub}} = -jb$$

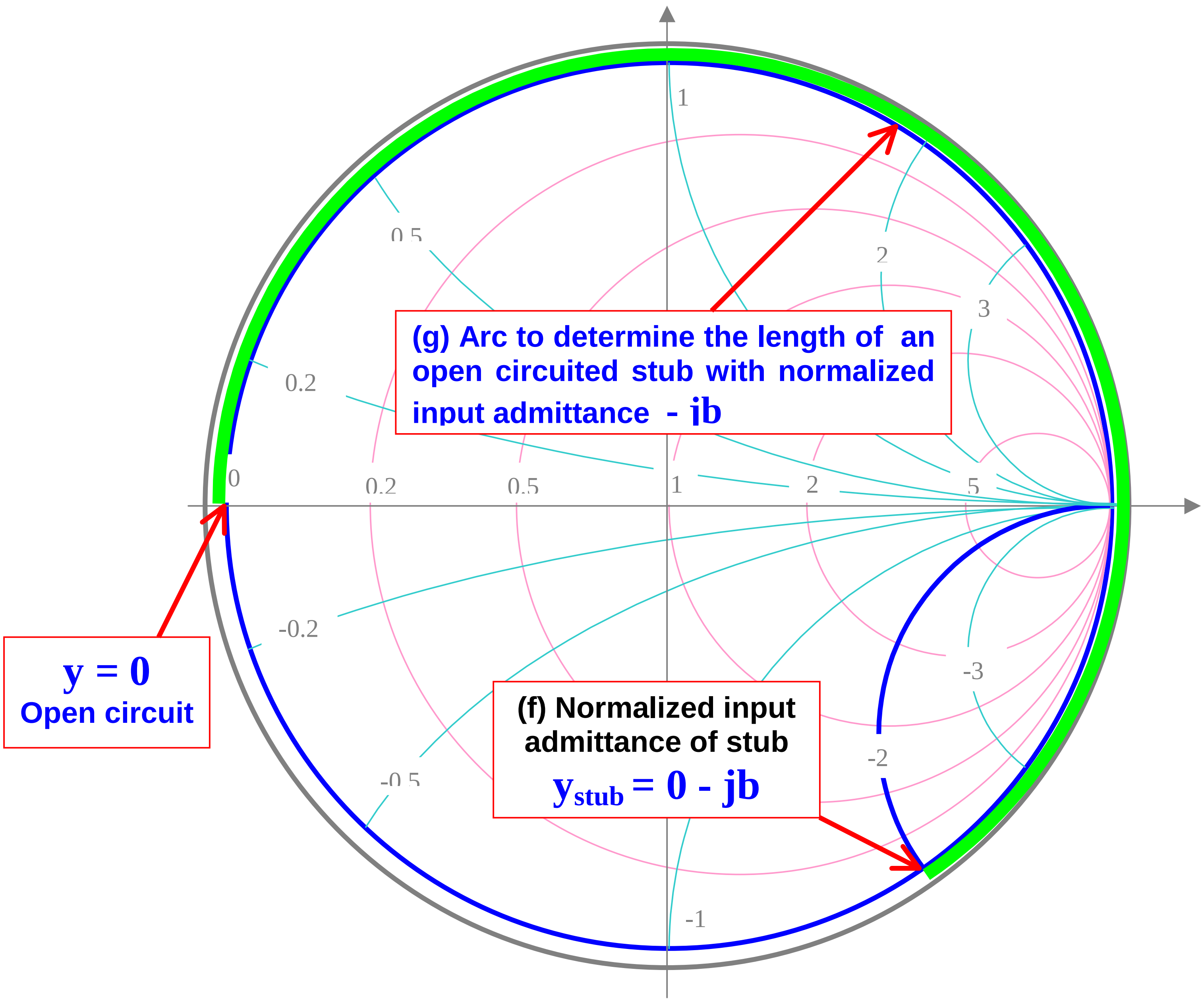
$$\text{line: } y(\mathrm{d_{stub}}) = 1 - jb \quad \rightarrow \quad \text{stub: } y_{\text{stub}} = +jb$$

(g) Use the chart to determine the length of the stub. The imaginary normalized admittance values are found on the circle of zero conductance on the chart. On a commercial Smith chart one can use a printed scale to read the stub length in terms of wavelength. We assume here that the stub line has characteristic impedance Z_0 as the main line. If the stub has characteristic impedance $Z_{0S} \neq Z_0$ the values on the Smith chart must be renormalized as

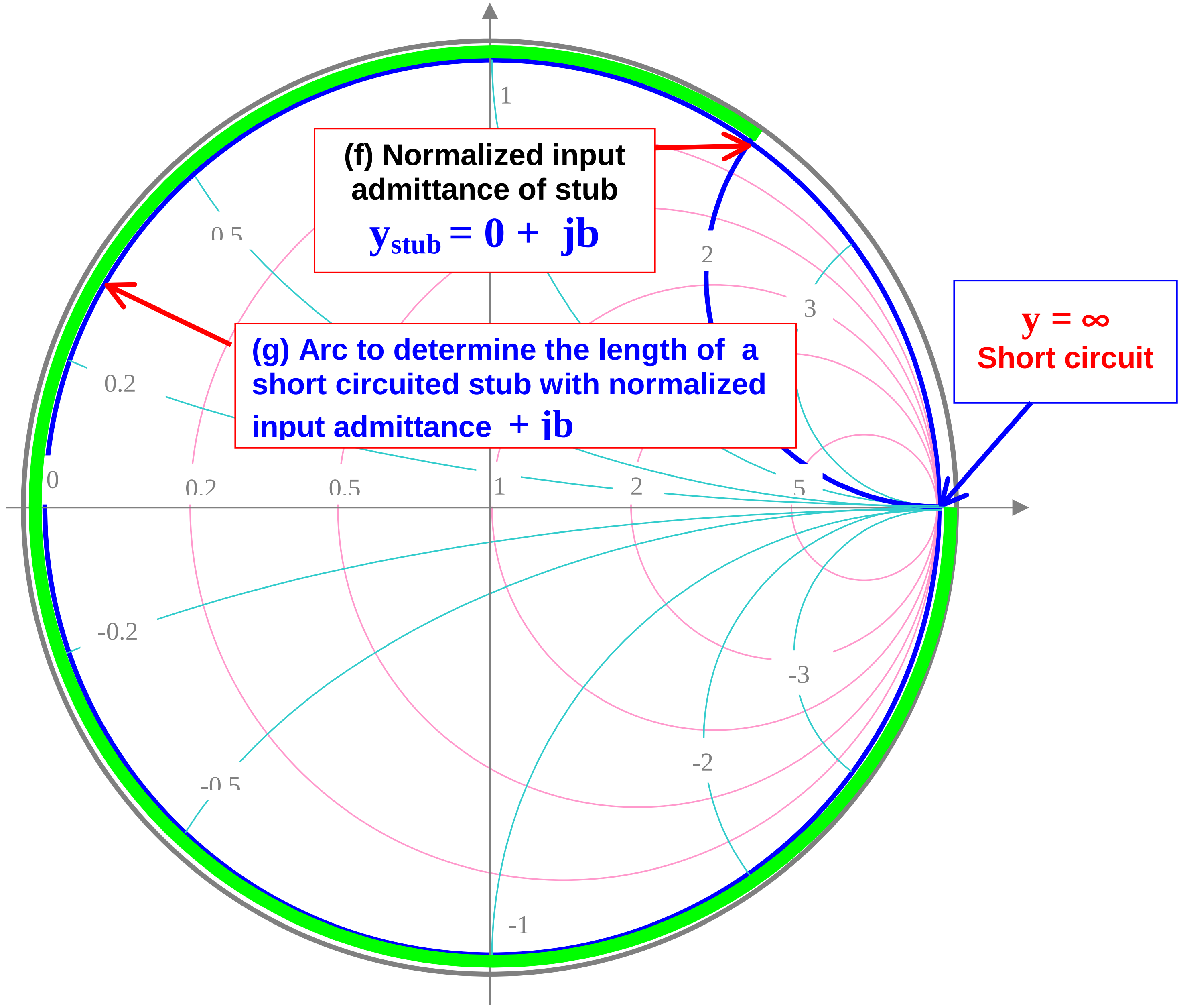
$$\pm jb' = \pm jb \frac{Y_0}{Y_{0s}} = \pm jb \frac{Z_{0s}}{Z_0}$$

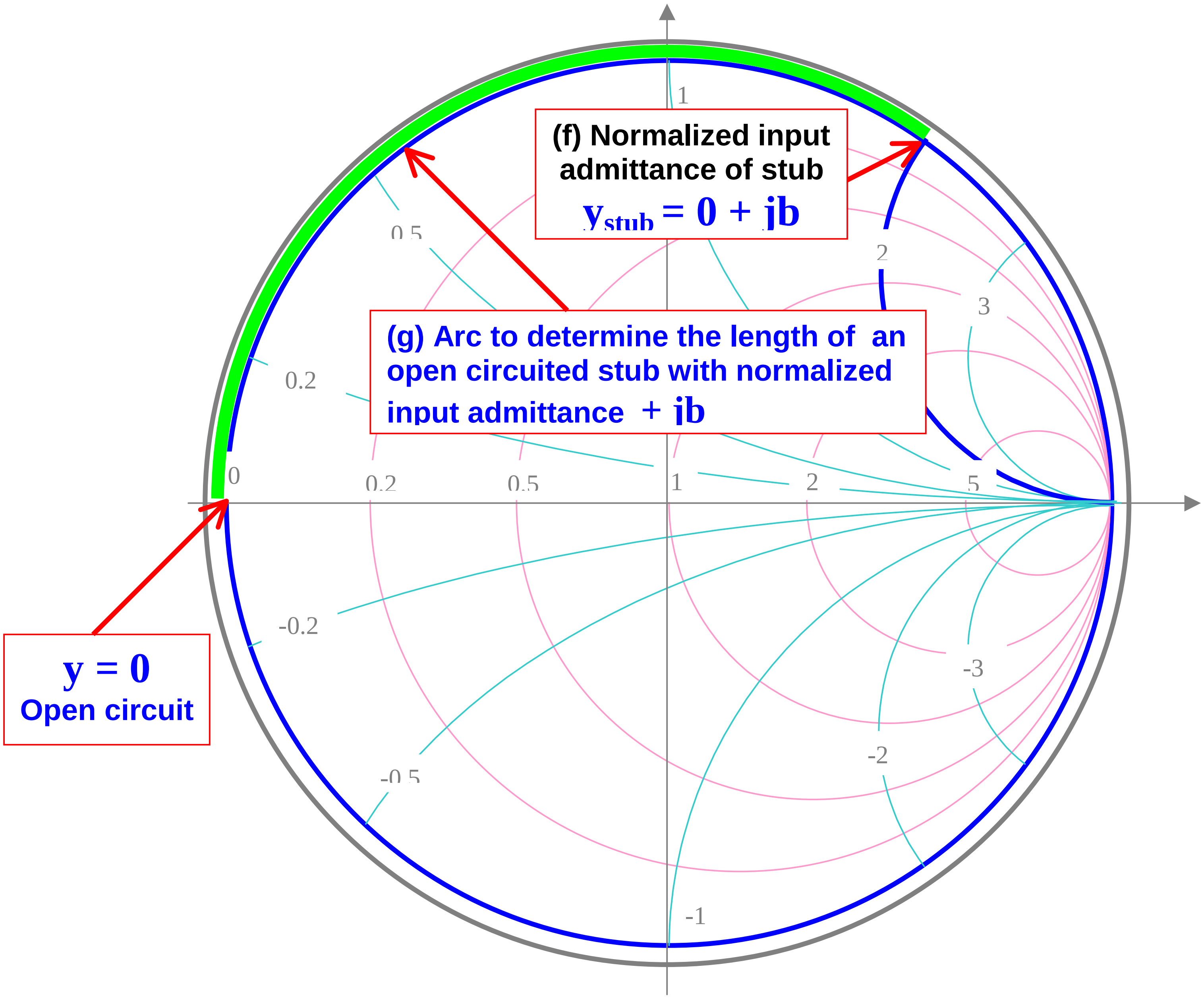


y = ∞
Short circuit
(g) Arc to determine the length of a short circuited stub with normalized input admittance - jb
(f) Normalized input admittance of stub
$y_{stub} = 0 - jb$
1
2
3
5
0.5
0.2
0
0.2
0.5
1
2
5
-0.2
-0.5
-1
-2
-3



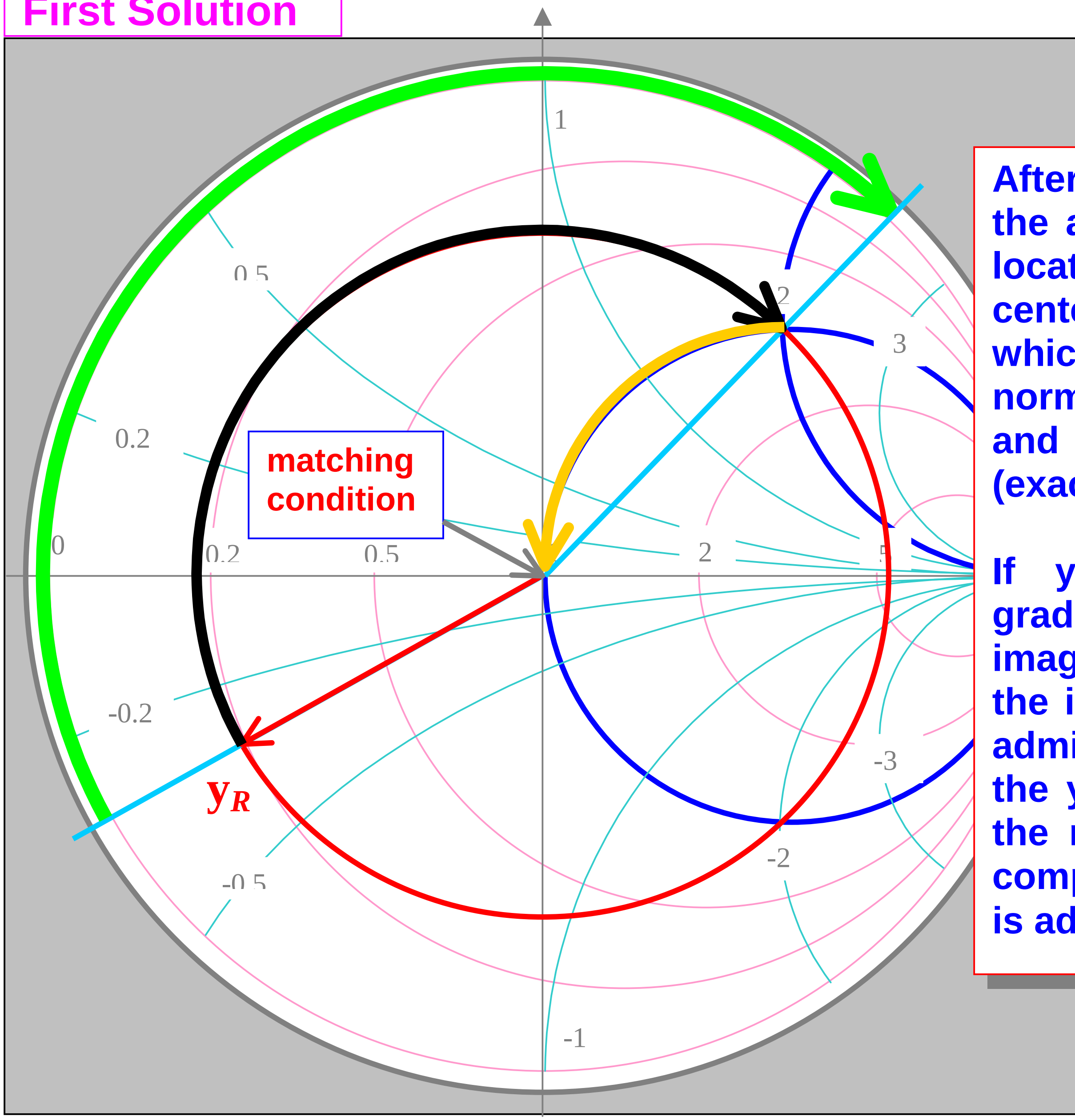
1
2
3
5
0
0.2
0.5
1
2
0.5
0.2
-0.2
-0.5
-1
-2
-3
(g) Arc to determine the length of an open circuited stub with normalized input admittance - jb
y = 0
Open circuit
(f) Normalized input admittance of stub
y_stub = 0 - jb


(f) Normalized input admittance of stub
$y_{stub} = 0 + jb$
(g) Arc to determine the length of a short circuited stub with normalized input admittance $+ jb$
$y = \infty$
Short circuit
1
0.5
2
3
0.2
0
0.2
0.5
1
2
5
-0.2
-3
-2
-0.5
-1



(f) Normalized input admittance of stub
$y_{stub} = 0 + jb$
(g) Arc to determine the length of an open circuited stub with normalized input admittance + jb
y = 0
Open circuit
0
0.2
0.5
1
2
5
3
-0.2
-0.5
-1
-2
-3

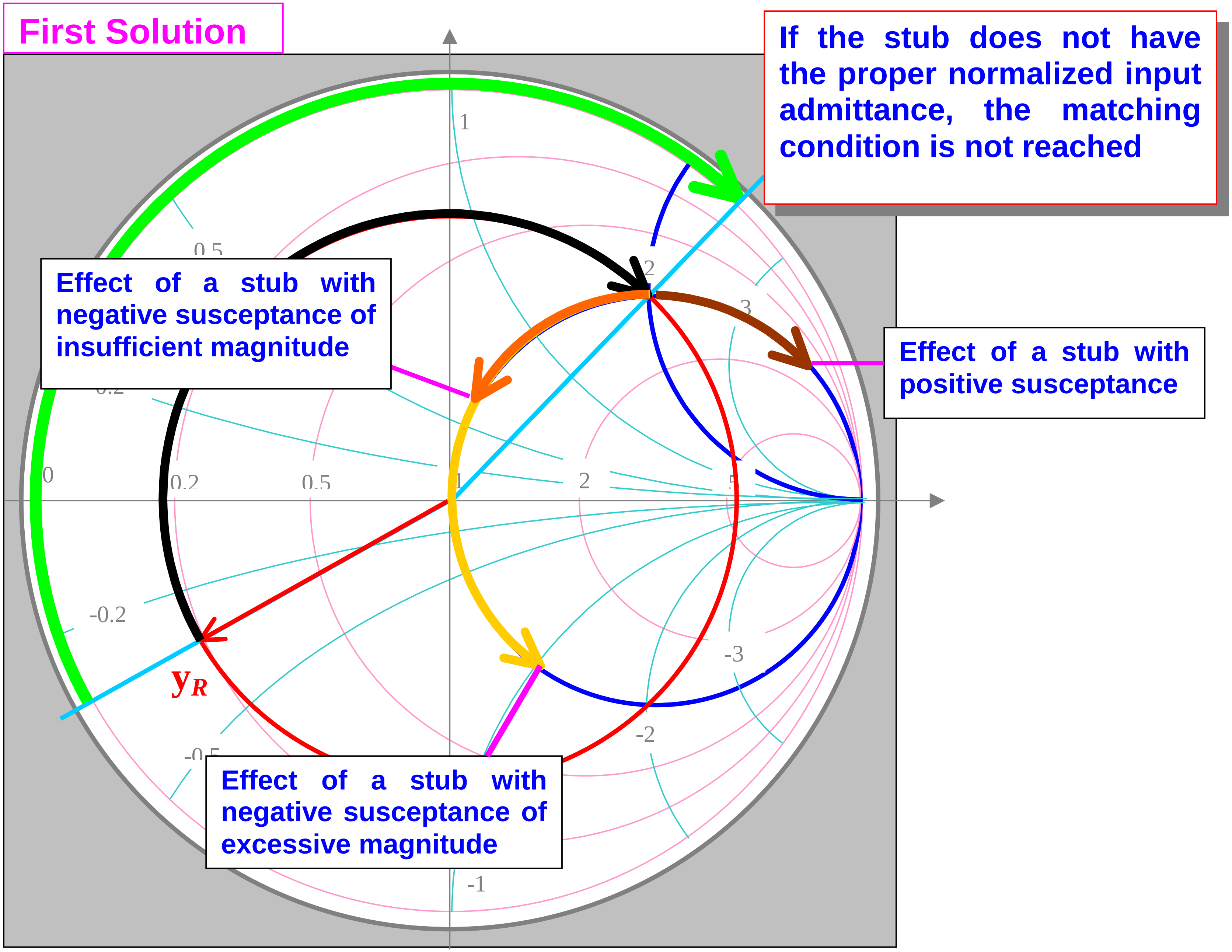
First Solution



After the stub is inserted, the admittance at the stub location is moved to the center of the Smith chart, which corresponds to normalized admittance 1 and reflection coefficient 0 (exact matching condition).

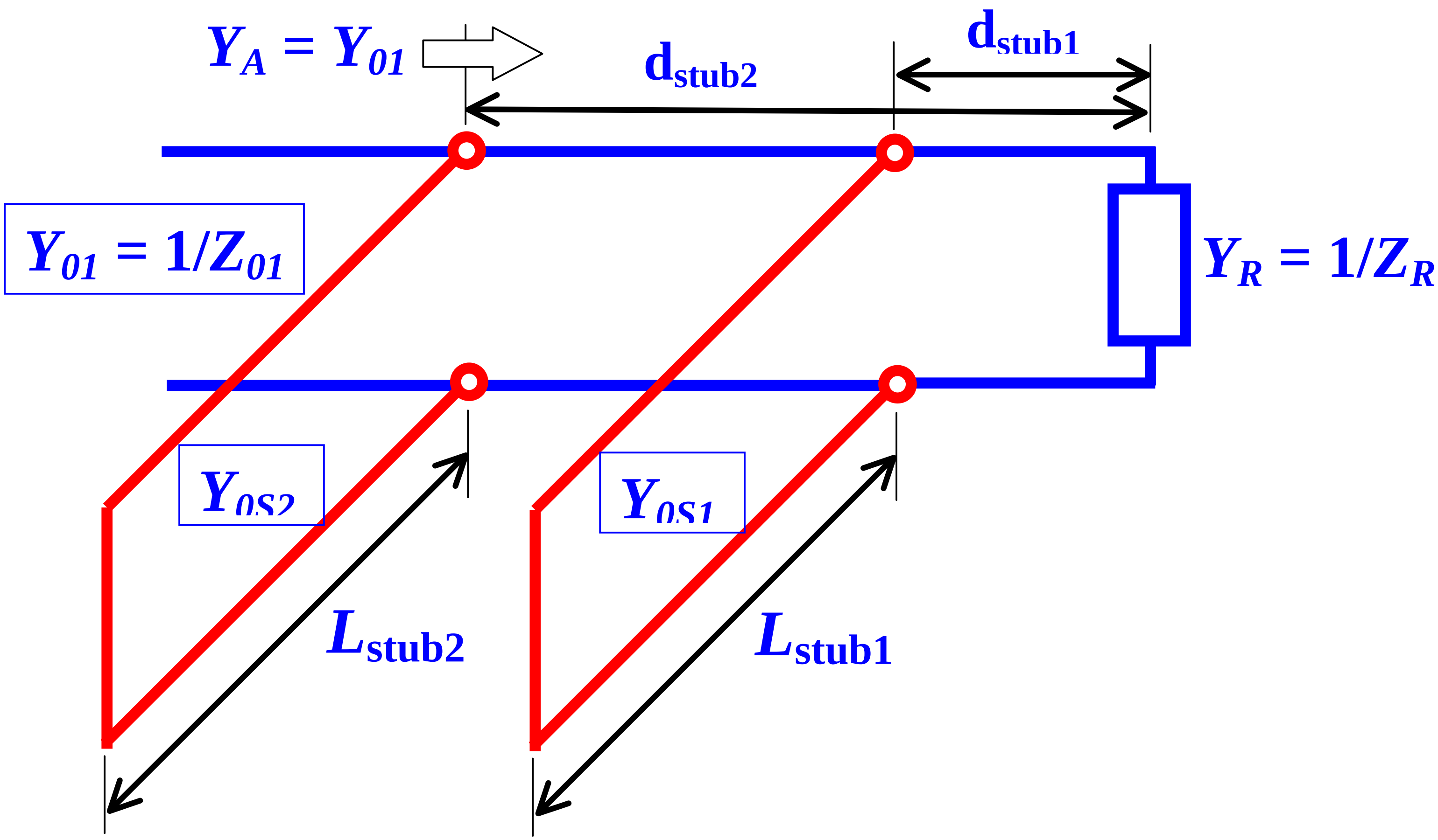
If you imagine to add gradually the negative imaginary admittance of the inserted stub, the total admittance would follow the yellow arrow, reaching the match point when the complete stub admittance is added.

First Solution



❑ Double stub impedance matching

Impedance matching can be achieved by inserting two stubs at specified locations along transmission line as shown below



There are two design parameters for double stub matching:

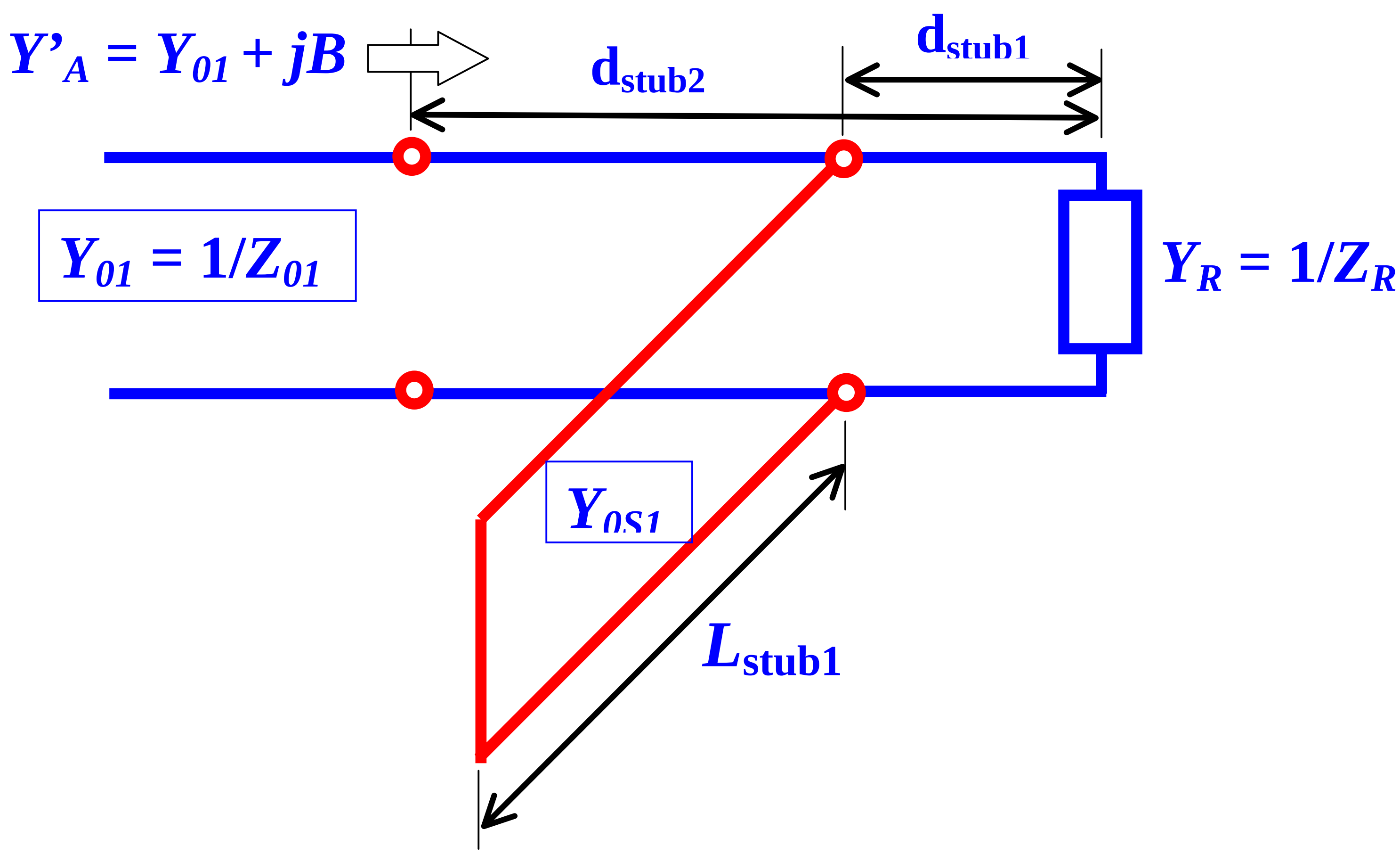
- **The length of the first stub line L_{stub1}**
- **The length of the second stub line L_{stub2}**

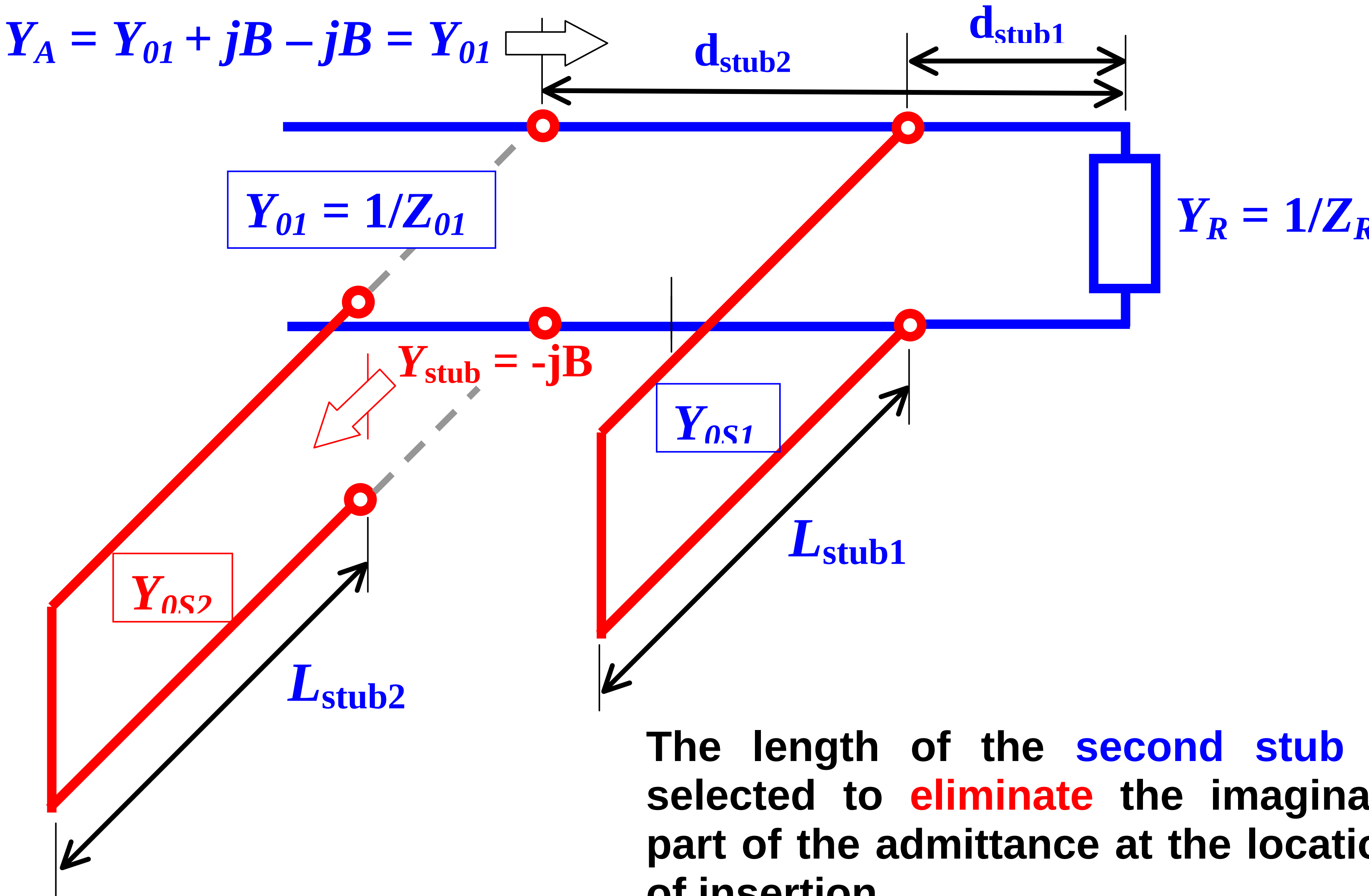
In the double stub configuration, the stubs are inserted at pre-determined locations. In this way, if the load impedance is changed, one simply has to replace the stubs with another set of different length.

The drawback of double stub tuning is that a certain range of load admittances cannot be matched once the stub locations are fixed.

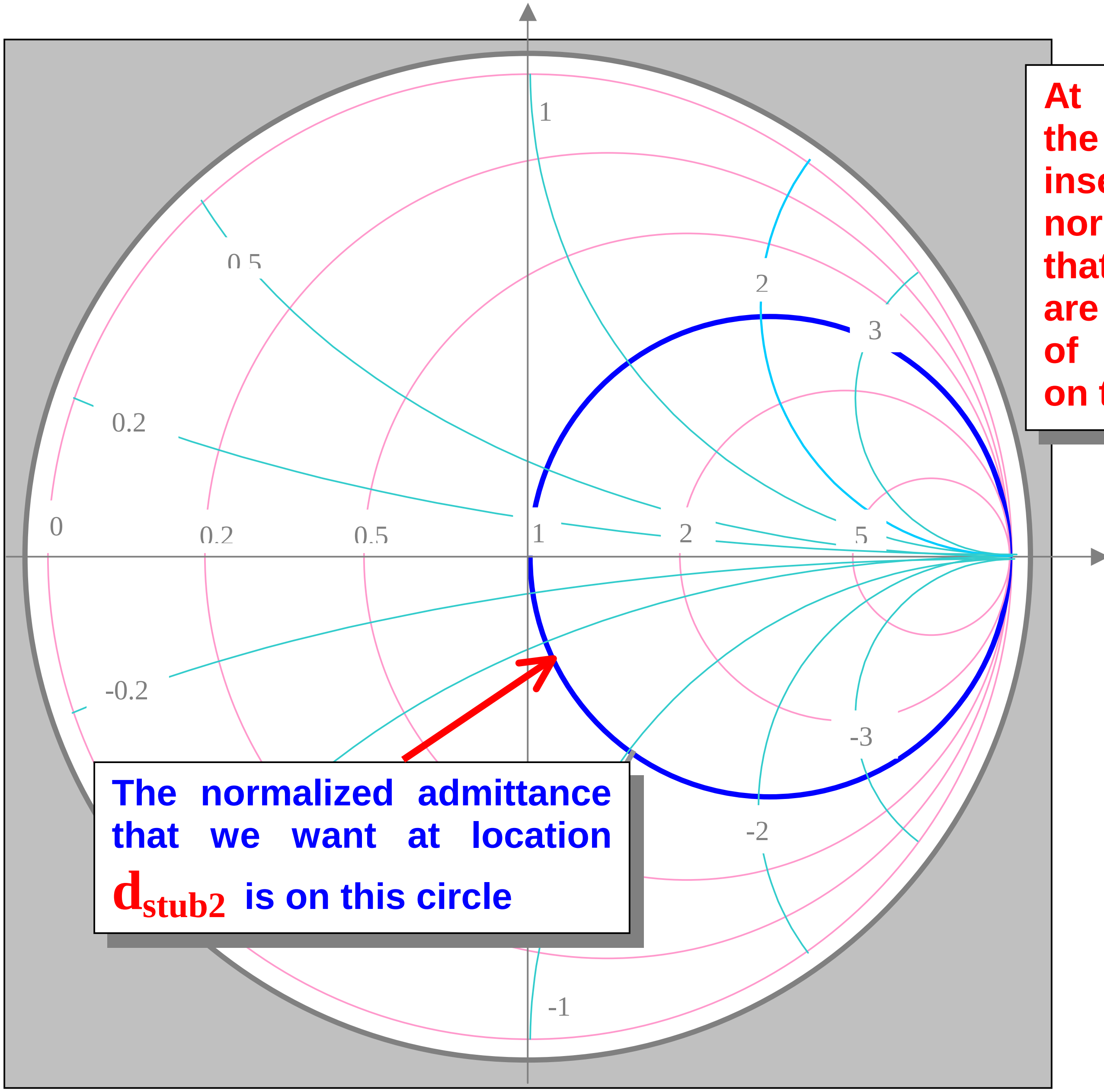
Three stubs are necessary to guarantee that match is always possible.

The length of the first stub is selected so that the admittance at the location of the second stub (before the second stub is inserted) has real part equal to the characteristic admittance of the line



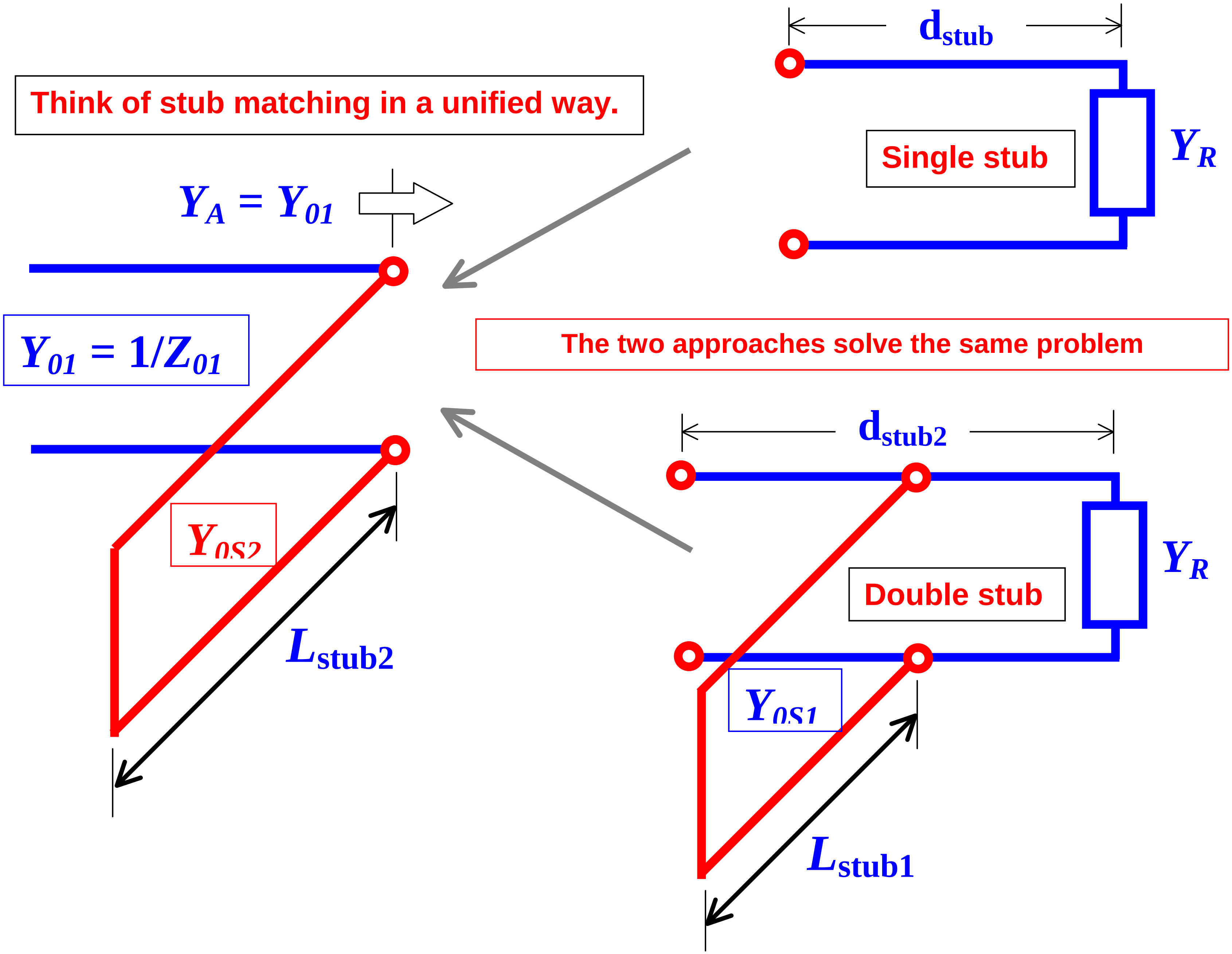


The length of the second stub is selected to eliminate the imaginary part of the admittance at the location of insertion.



At the location where the second stub is inserted, the possible normalized admittances that can give matching are found on the circle of unitary conductance on the Smith chart.

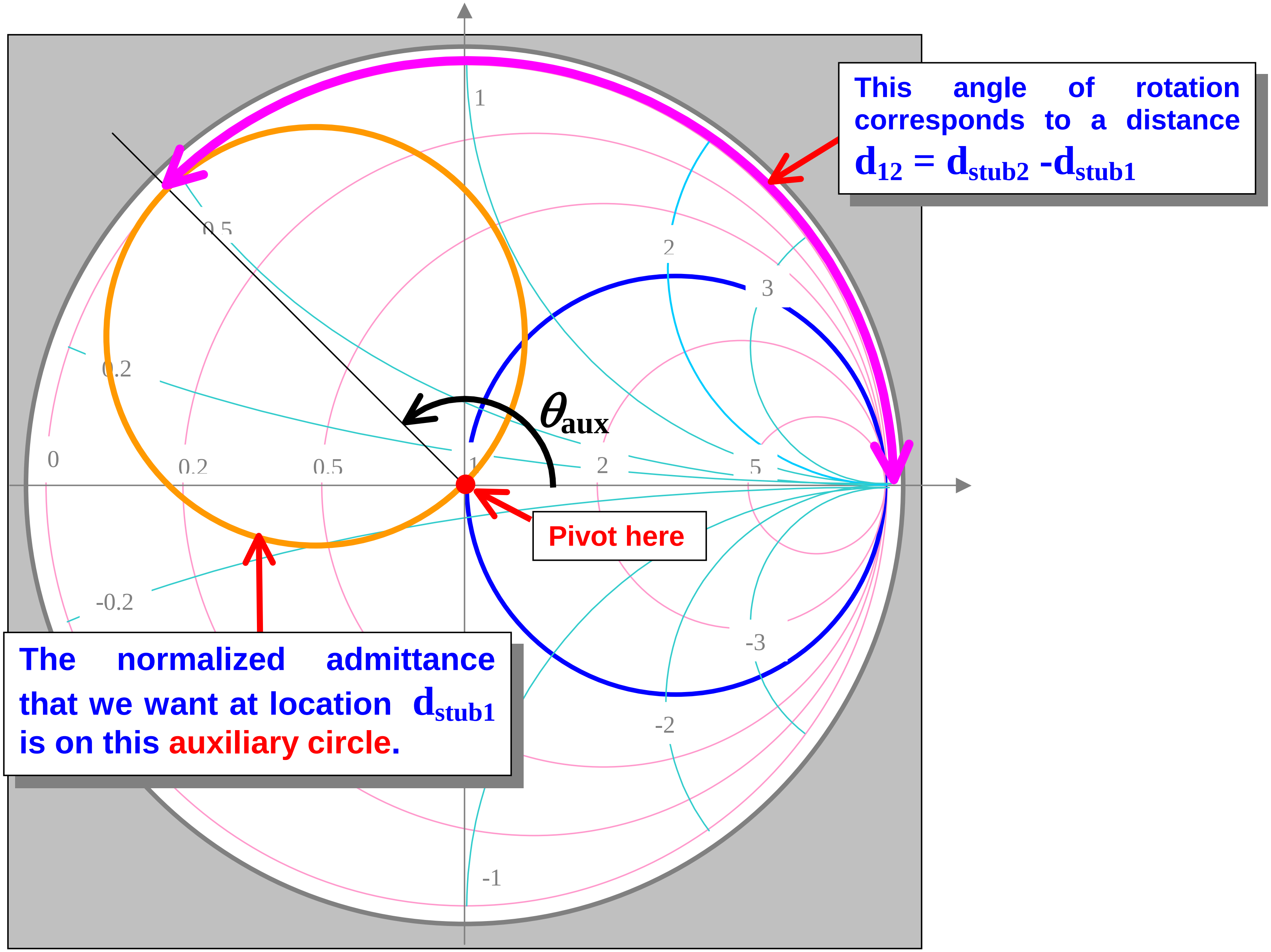
The normalized admittance that we want at location d_{stub2} is on this circle


Think of stub matching in a unified way.
$Y_A = Y_{01}$
$Y_{01} = 1/Z_{01}$
Y_{0S2}
L_{stub2}
d_{stub}
Single stub
Y_R
The two approaches solve the same problem
d_{stub2}
Double stub
Y_R
Y_{0S1}
L_{stub1}

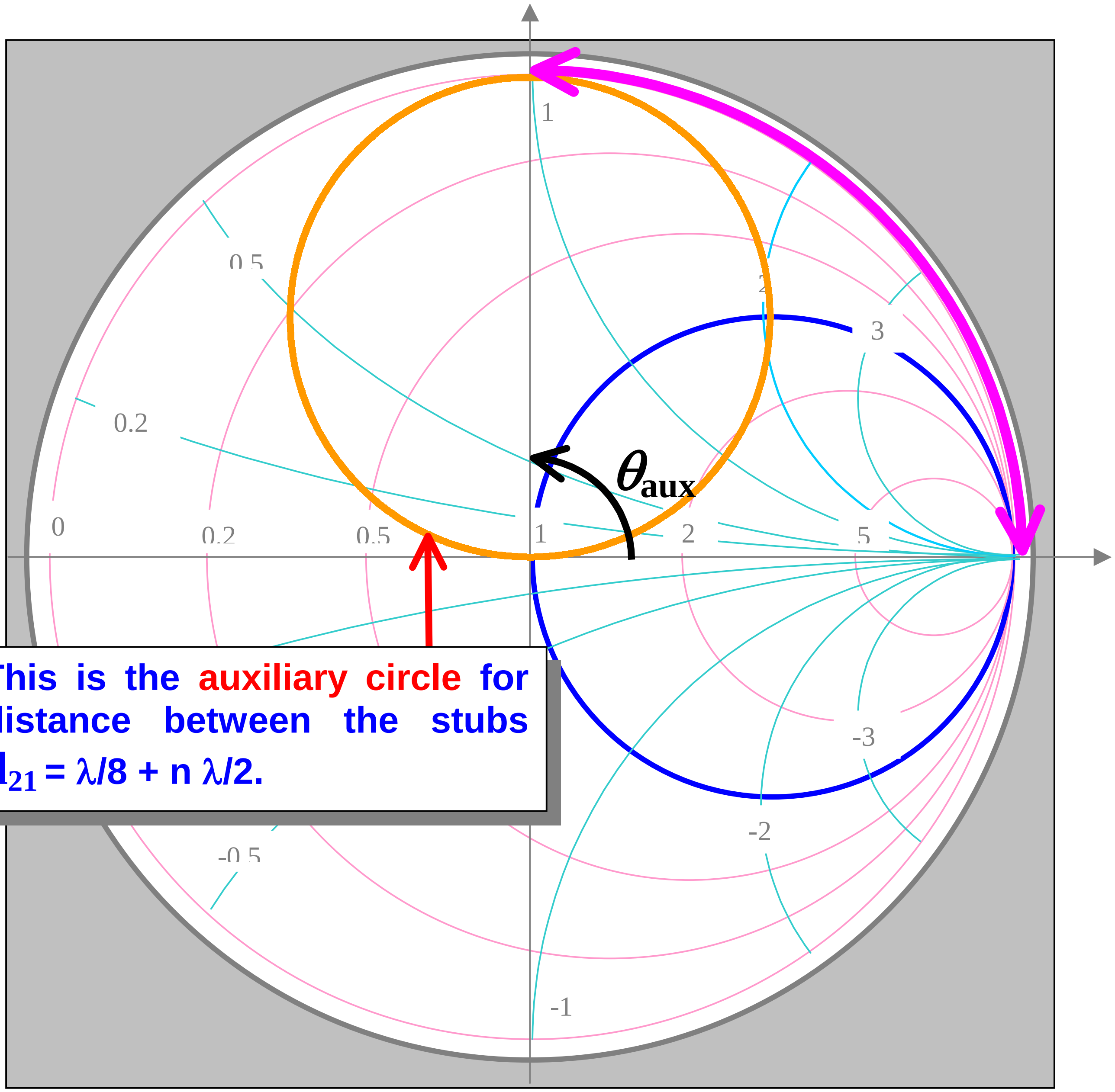
If one moves from the location of the second stub back to the load, the circle of the allowed normalized admittances is mapped into another circle, obtained by pivoting the original circle about the center of the chart.

At the location of the first stub, the allowed normalized admittances are found on an auxiliary circle which is obtained by rotating the unitary conductance circle counterclockwise, by an angle

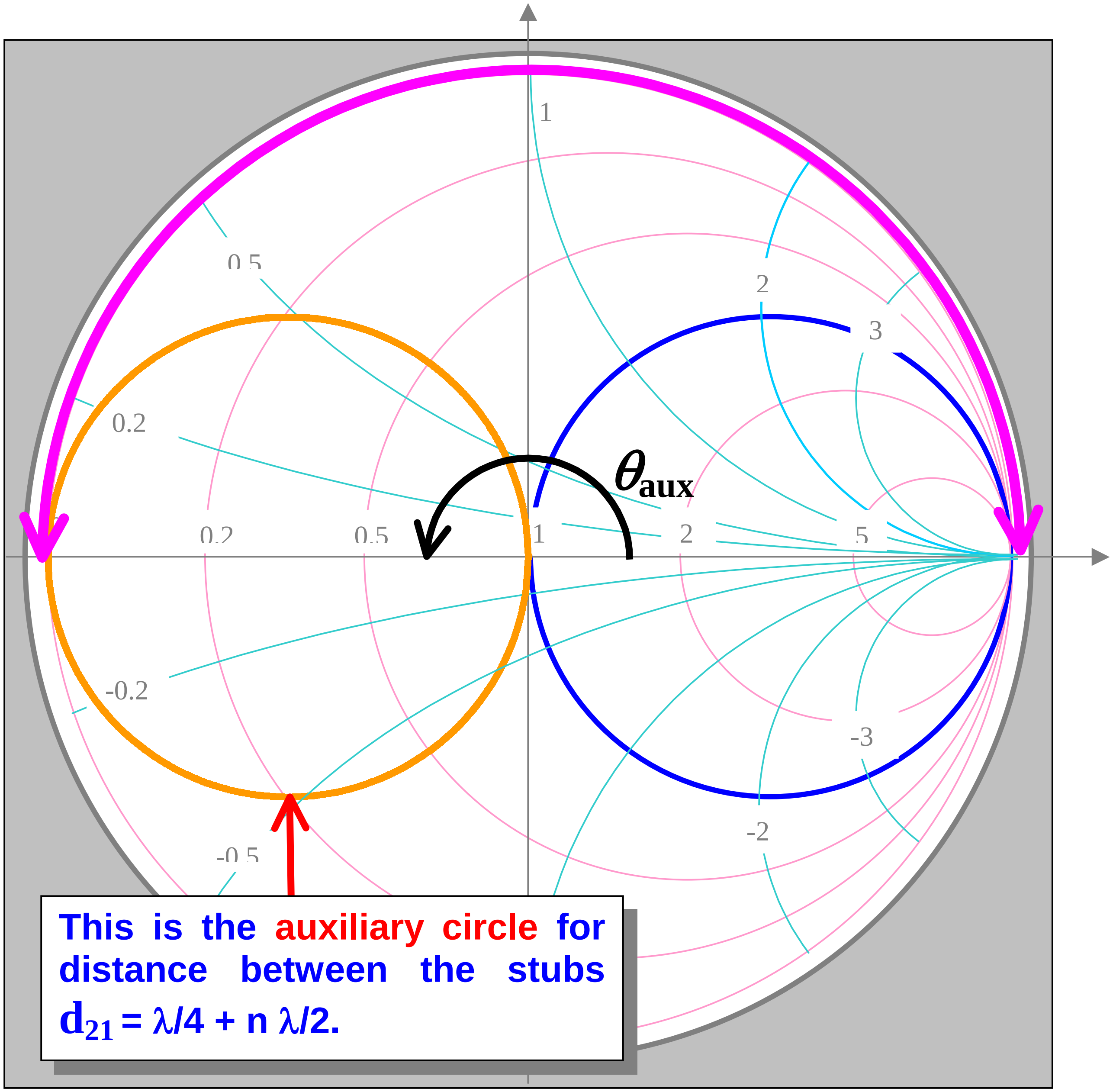
$$\theta_{aux} = \frac{4\pi}{\lambda}\left(d_{stub2} - d_{stub1}\right) = \frac{4\pi}{\lambda} d_{21}$$

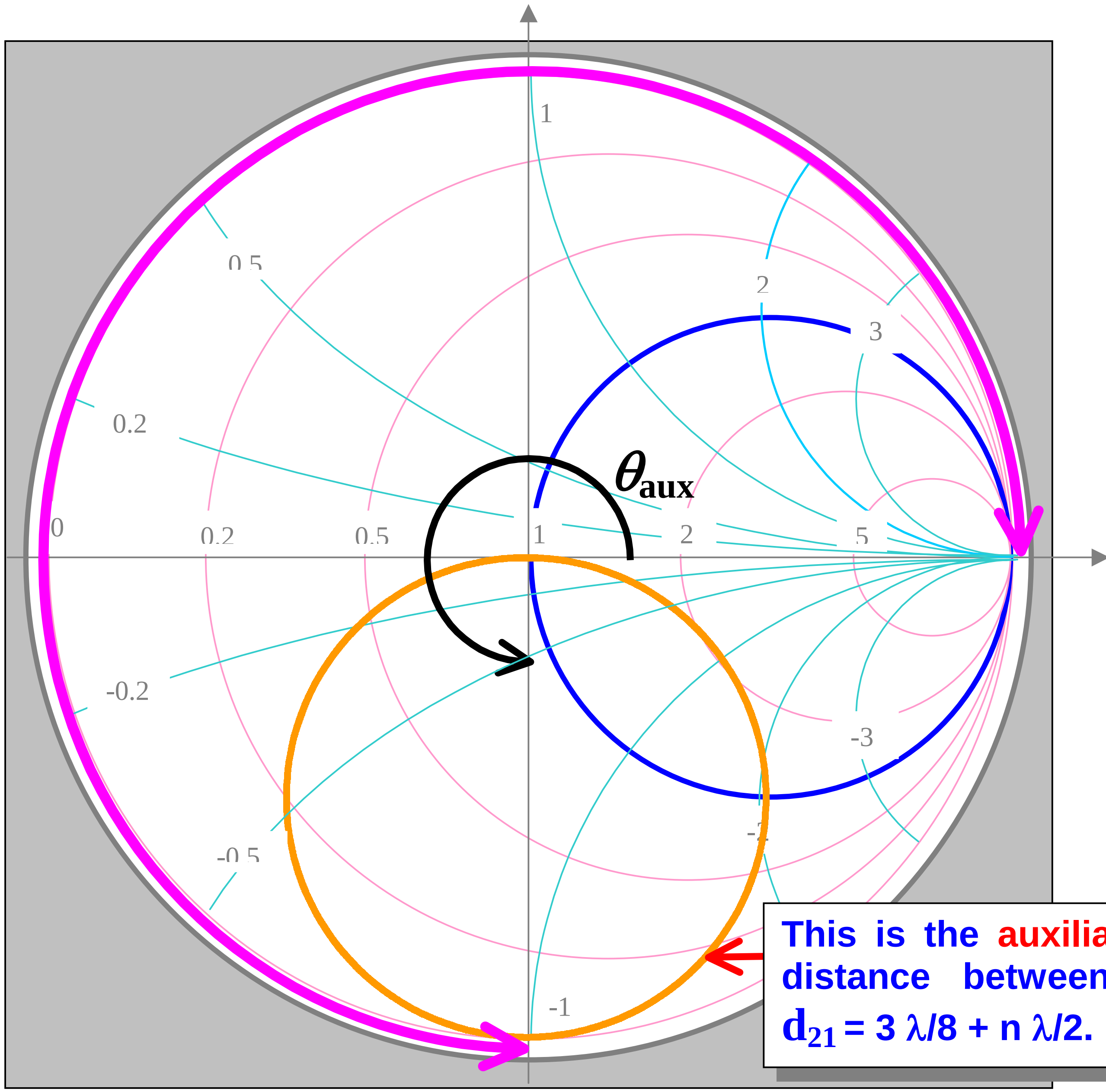


This angle of rotation corresponds to a distance
$d_{12} = d_{stub2} - d_{stub1}$
θ_{aux}
Pivot here
The normalized admittance that we want at location d_{stub1} is on this auxiliary circle.
1
0.5
0.2
0
0.2
0.5
1
2
5
2
3
-0.2
-3
-2
-1

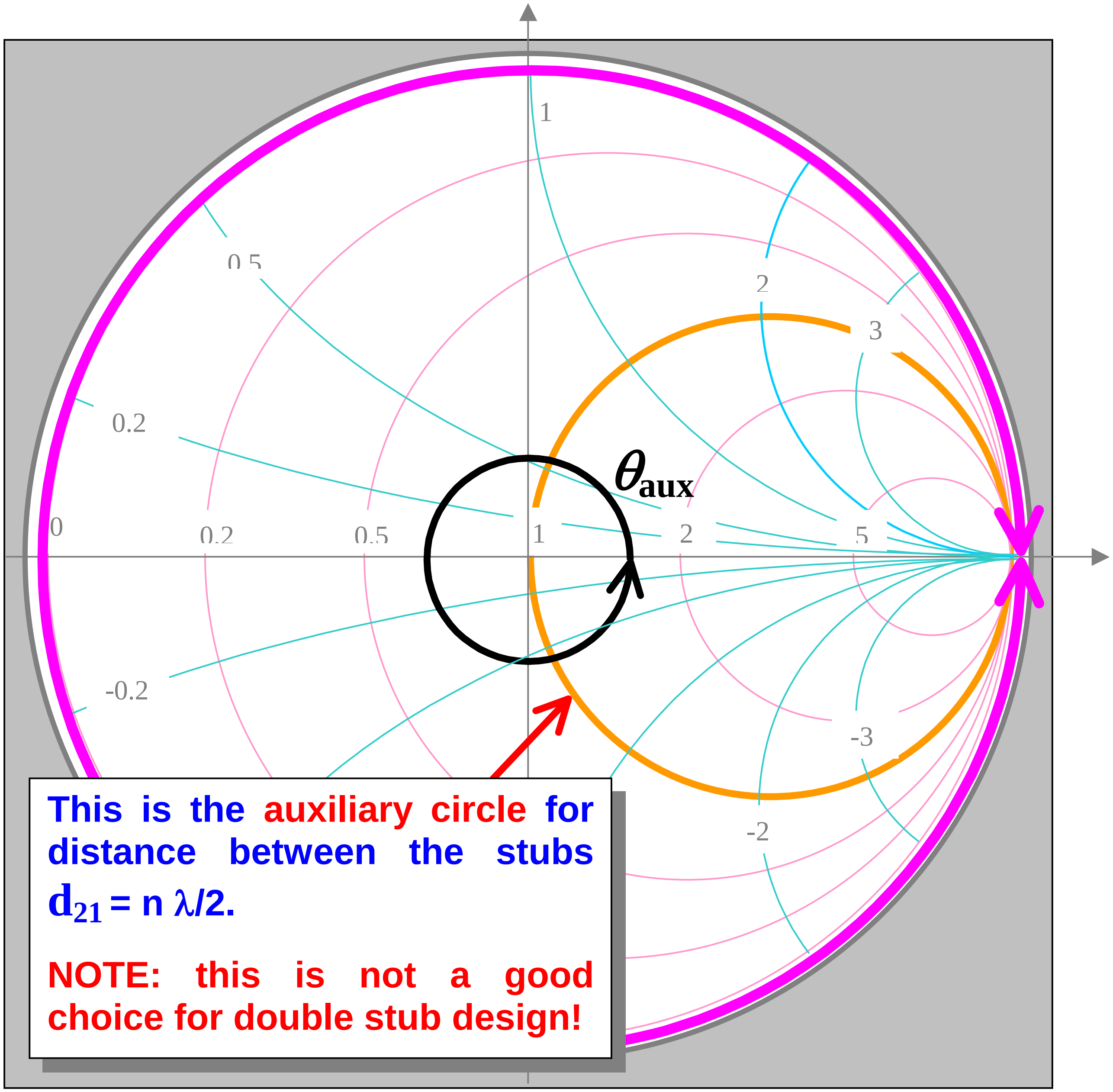


This is the **auxiliary circle** for distance between the stubs $d_{21} = \lambda/8 + n\,\lambda/2$.


1
0.5
2
3
0.2
0.2
0.5
1
2
5
θ_{aux}
-0.2
-3
-2
-0.5
This is the auxiliary circle for distance between the stubs $d_{21} = \lambda/4 + n\,\lambda/2$.



This is the auxiliary circle for distance between the stubs $d_{21} = 3\ \lambda/8 + n\ \lambda/2$.



This is the auxiliary circle for distance between the stubs $d_{21} = n\,\lambda/2$.

NOTE: this is not a good choice for double stub design!

Given the load impedance, we need to follow these steps to complete the double stub design:

(a) Find the normalized load impedance and determine the corresponding location on the chart.

(b) Draw the circle of constant magnitude of the reflection coefficient |Γ| for the given load.

(c) Determine the normalized load admittance on the chart. This is obtained by rotating -180° on the constant |Γ| circle, from the load impedance point. From now on, all values read on the chart are normalized admittances.

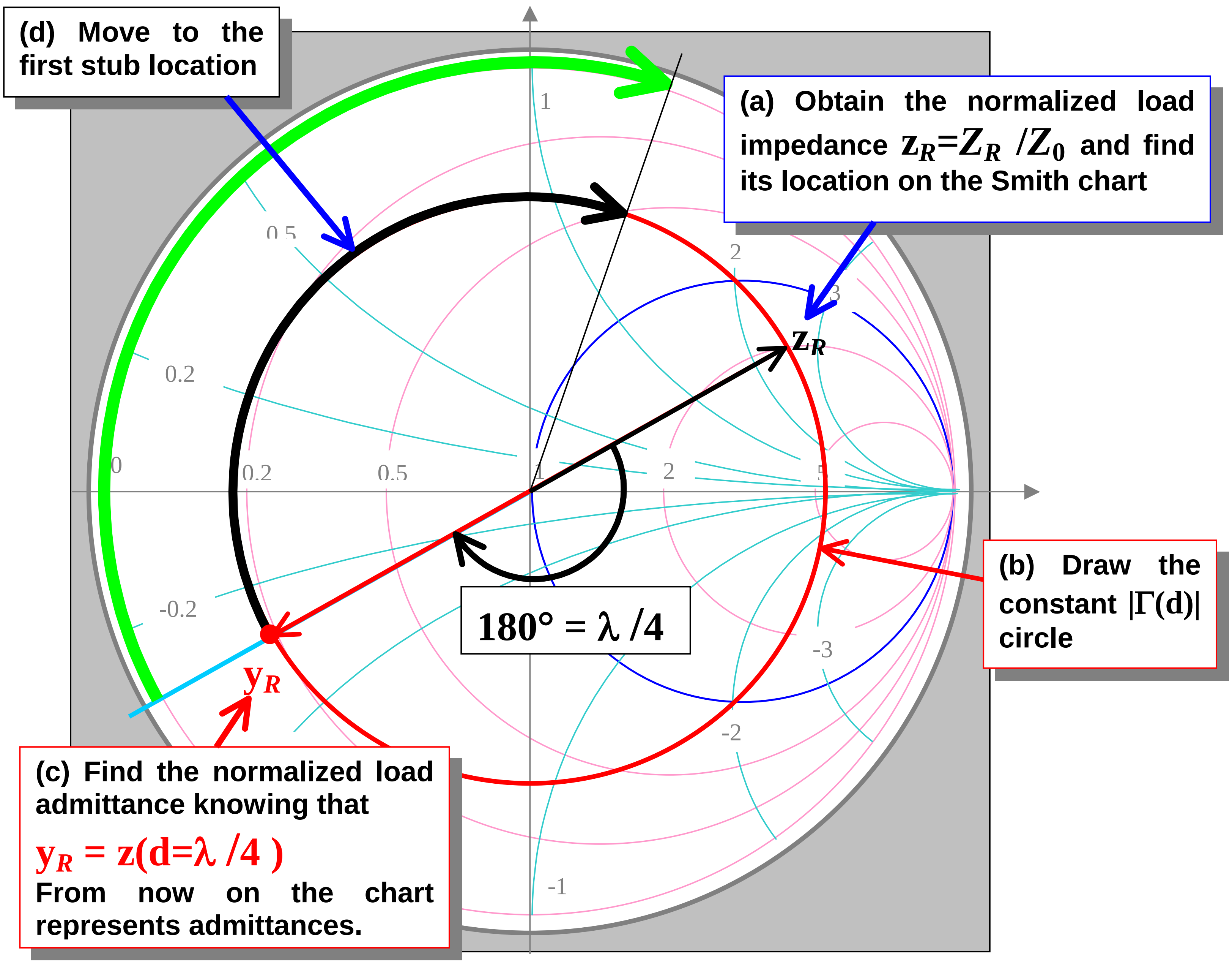
(d) Find the normalized admittance at location d_{stub1} by moving clockwise on the constant |Γ| circle.

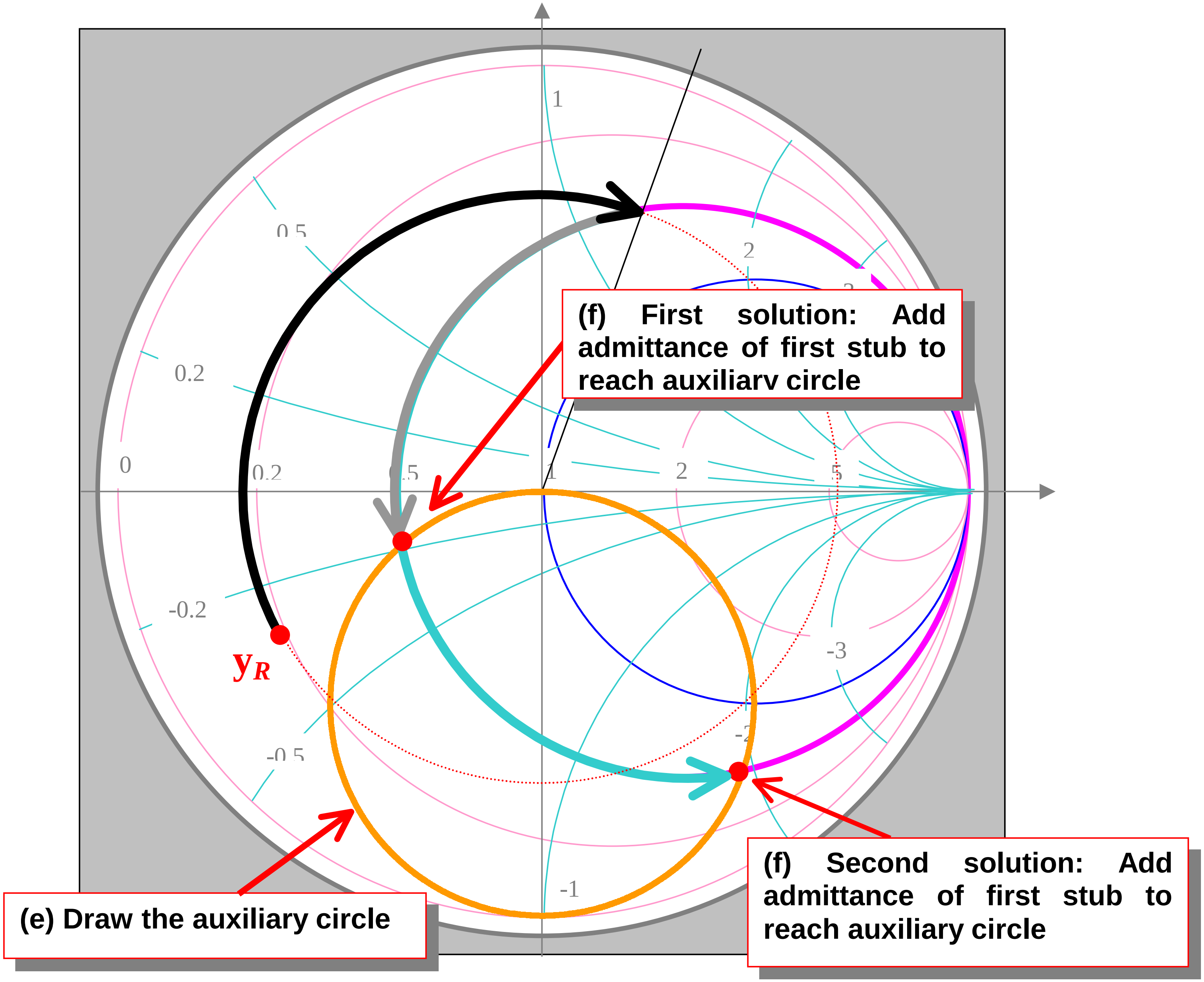
(e) Draw the auxiliary circle

(f) Add the first stub admittance so that the normalized admittance point on the Smith chart reaches the auxiliary circle (two possible solutions). The admittance point will move on the corresponding conductance circle, since the stub does not alter the real part of the admittance

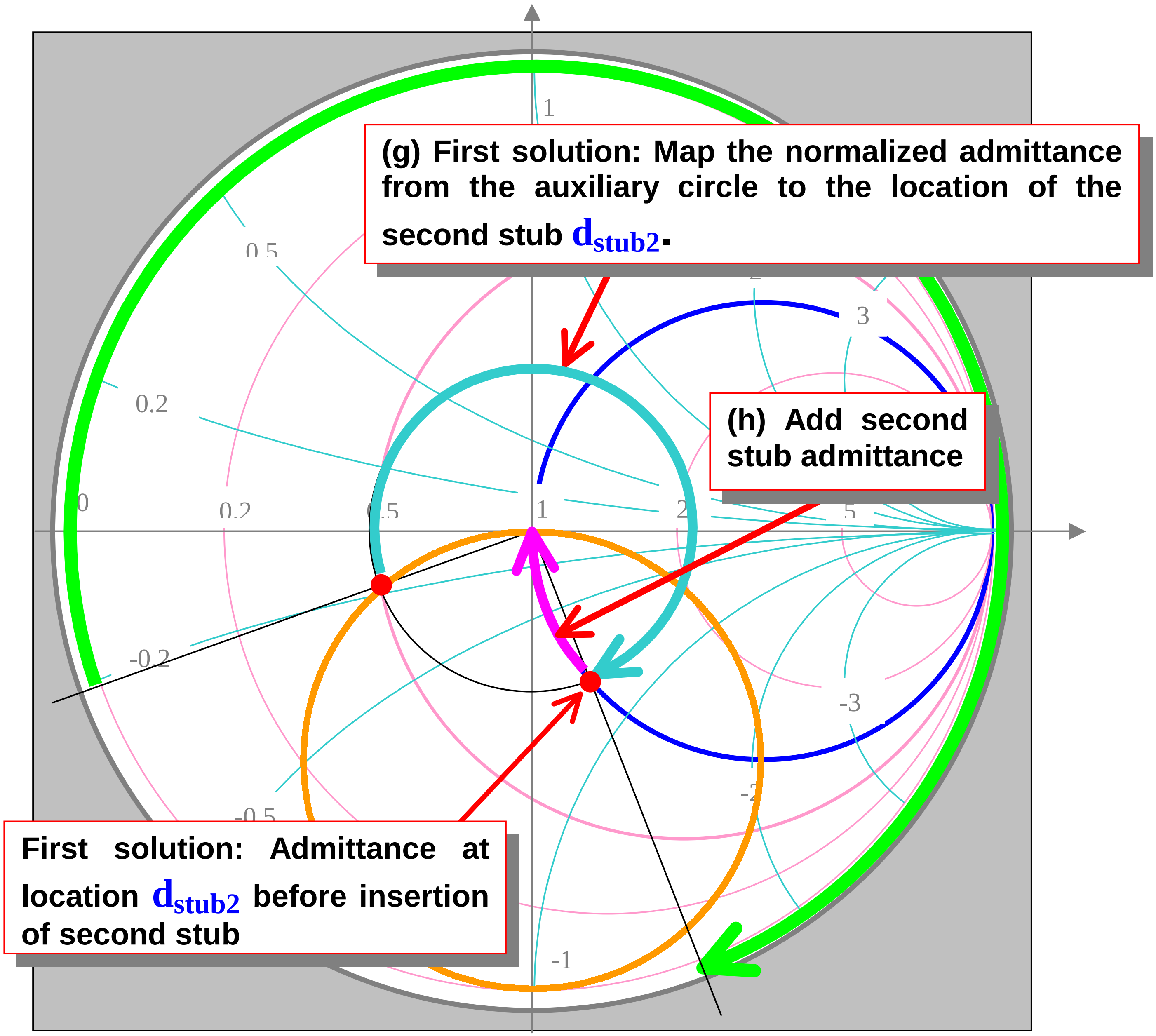
(g) Map the normalized admittance obtained on the auxiliary circle to the location of the second stub d_{stub2}. The point must be on the unitary conductance circle

(h) Add the second stub admittance so that the total parallel admittance equals the characteristic admittance of the line to achieve exact matching condition

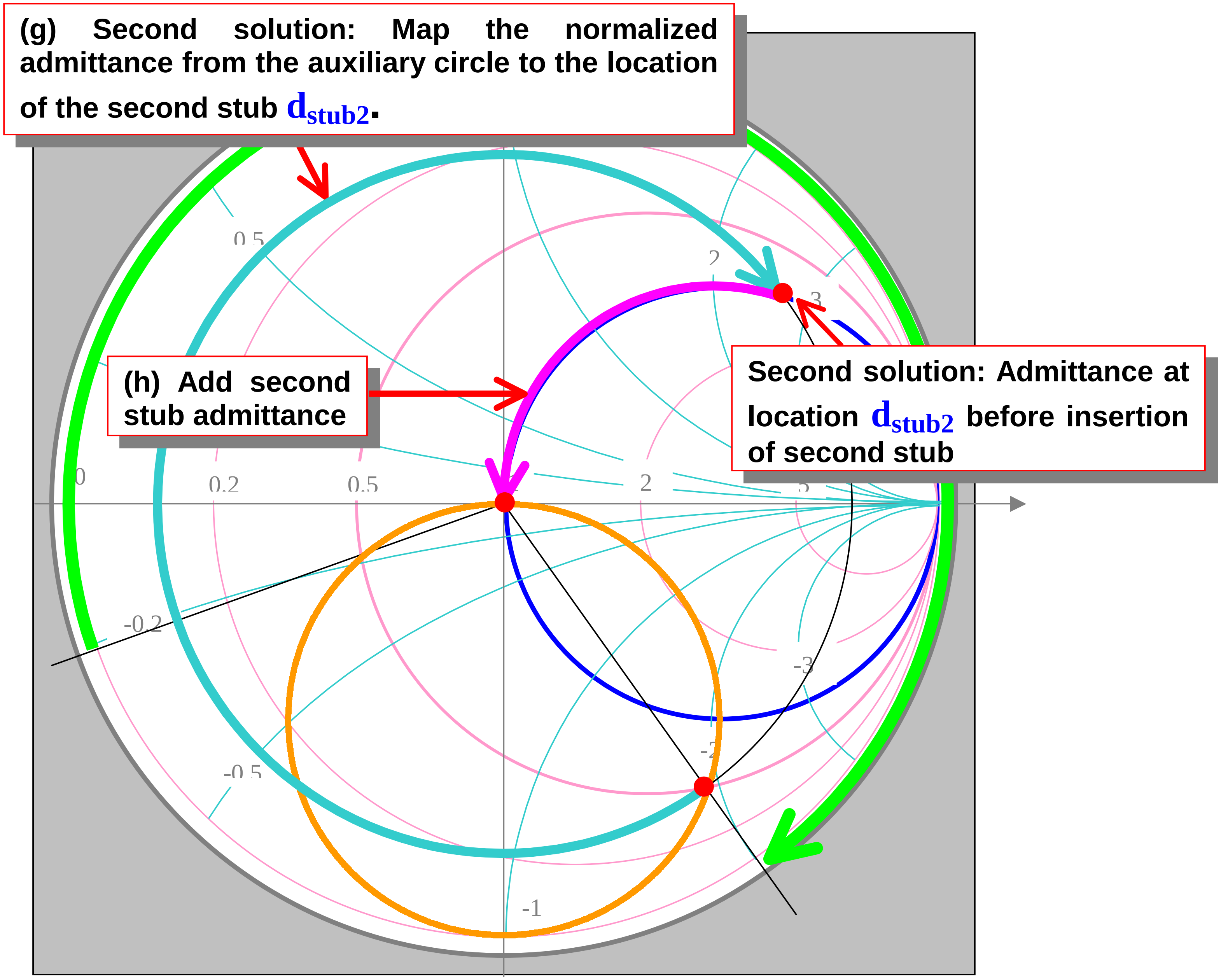

(d) Move to the first stub location
(a) Obtain the normalized load impedance $z_R = Z_R / Z_0$ and find its location on the Smith chart
z_R
(b) Draw the constant $|\Gamma(d)|$ circle
180° = λ /4
y_R
(c) Find the normalized load admittance knowing that
$y_R = z(d=\lambda/4)$
From now on the chart represents admittances.
1
0.5
0.2
0
0.2
0.5
1
2
5
-0.2
-3
-2
-1
2
3



1
0.5
2
0.2
0
0.2
0.5
1
2
5
-0.2
y_R
-3
-0.5
-1
(f) First solution: Add admittance of first stub to reach auxiliary circle
(e) Draw the auxiliary circle
(f) Second solution: Add admittance of first stub to reach auxiliary circle



(g) First solution: Map the normalized admittance from the auxiliary circle to the location of the second stub d_{stub2}.
(h) Add second stub admittance
First solution: Admittance at location d_{stub2} before insertion of second stub
1
0.5
0.2
0
0.2
0.5
1
2
3
5
-0.2
-0.5
-1
-2
-3

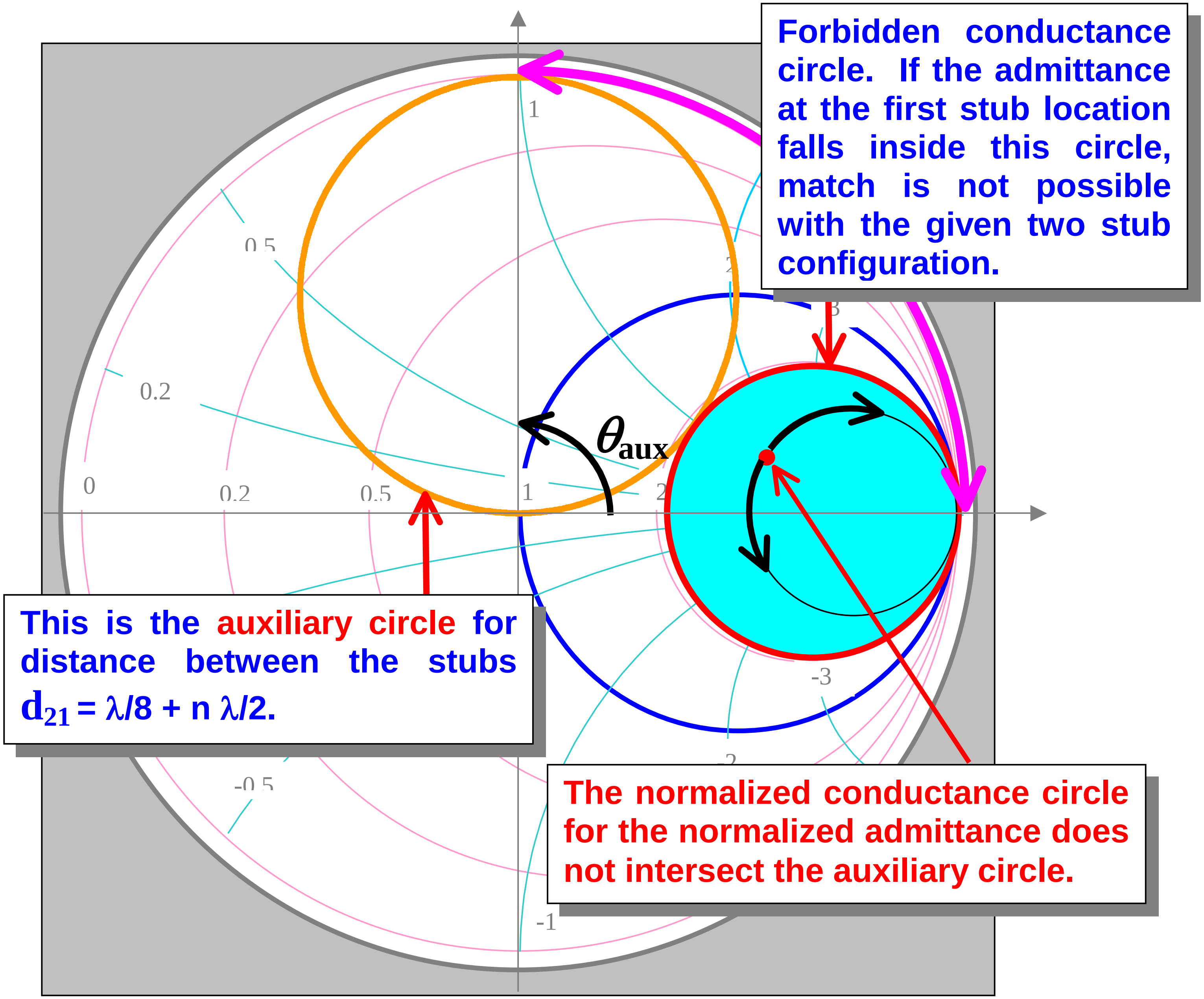

(g) Second solution: Map the normalized admittance from the auxiliary circle to the location of the second stub d_{stub2}.
(h) Add second stub admittance
Second solution: Admittance at location d_{stub2} before insertion of second stub
0.5
2
3
0
0.2
0.5
2
-0.2
-3
-0.5
-2
-1

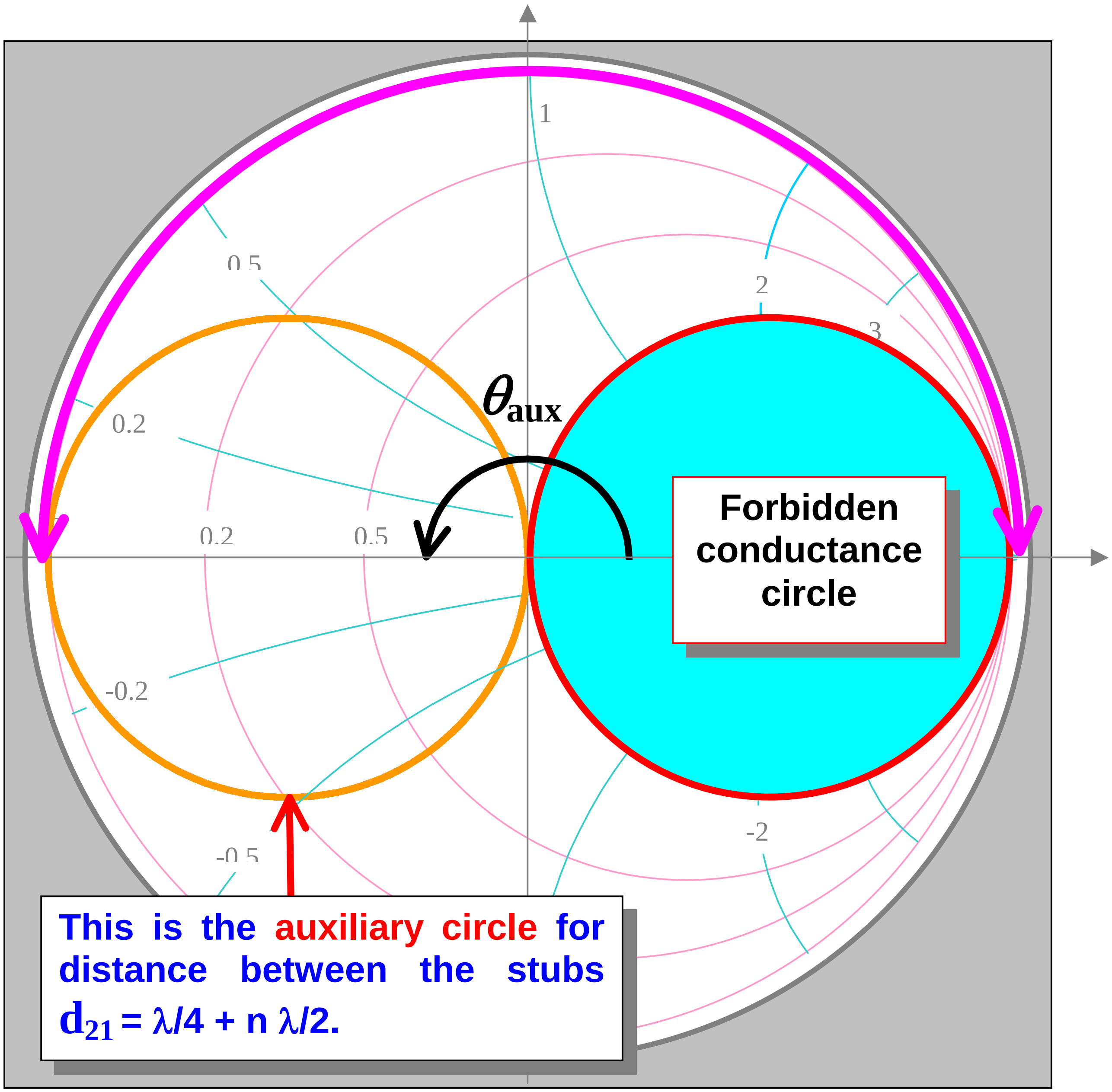
As mentioned earlier, a double stub configuration with fixed stub location may not be able to match a certain range of load impedances.

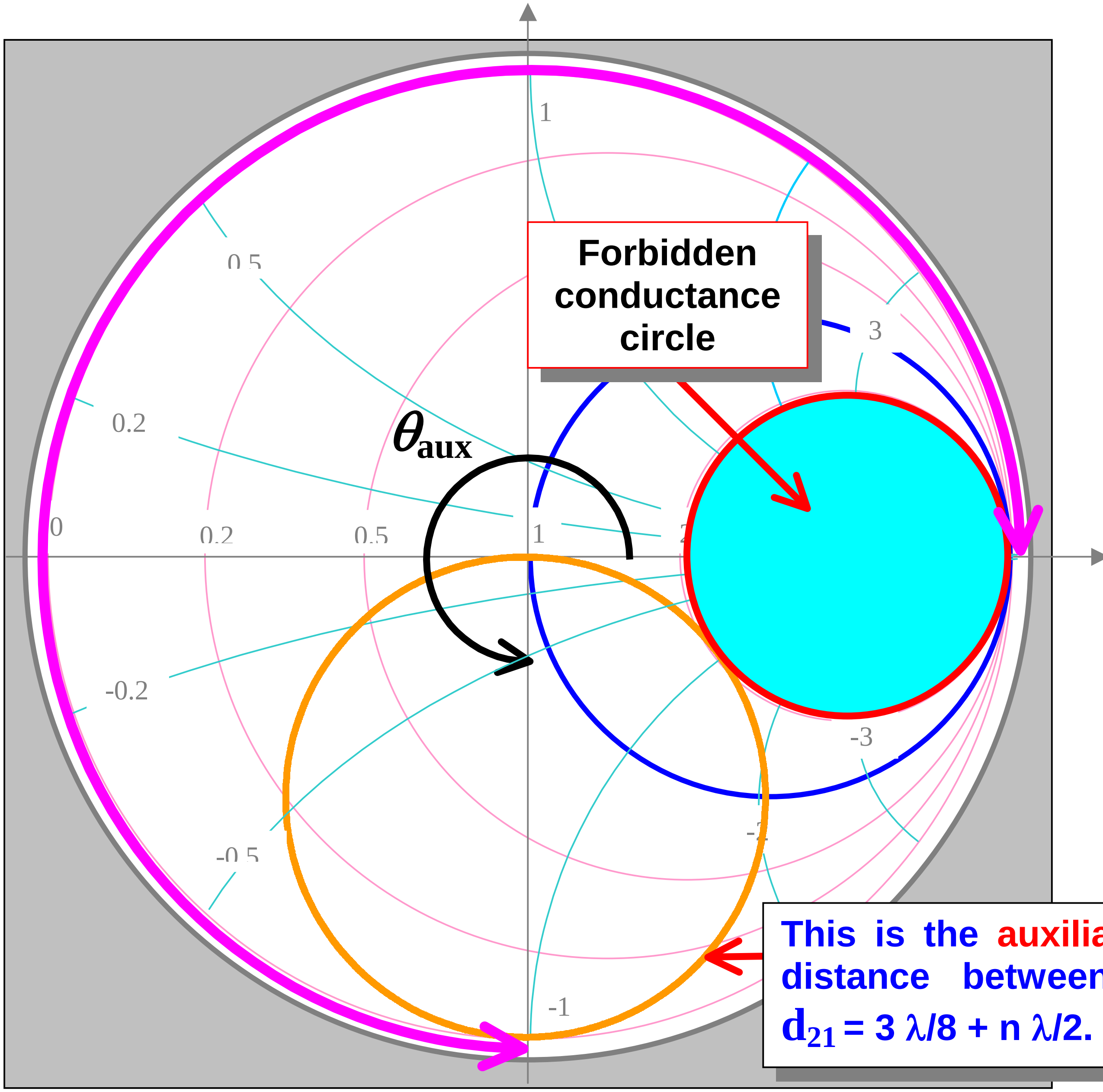
This is easily seen on the Smith chart. If the normalized admittance of the line, at the first stub location, falls inside a certain forbidden conductance circle tangent to the auxiliary circle (and always contained inside the unitary conductance circle), it is not possible to find a value for the first stub that can bring the normalized admittance to the auxiliary circle. Therefore, it is impossible to position the normalized admittance of the second stub location on the unitary conductance circle.

When this condition occurs, the location of one of the stubs must be changed appropriately. Alternatively, a third stub could be added.

Examples of forbidden regions follow.


Forbidden conductance circle. If the admittance at the first stub location falls inside this circle, match is not possible with the given two stub configuration.
θaux
This is the auxiliary circle for distance between the stubs d21 = λ/8 + n λ/2.
The normalized conductance circle for the normalized admittance does not intersect the auxiliary circle.


1
0.5
2
3
0.2
0.2
0.5
-0.2
-0.5
-2
θ_{aux}
Forbidden conductance circle
This is the auxiliary circle for distance between the stubs $d_{21} = \lambda/4 + n\ \lambda/2$.



This is the auxiliary circle for distance between the stubs $d_{21} = 3\lambda/8 + n\lambda/2$.